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BY
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ABSTRACT

This investigation deals with the use of a movable mass control system to stabilize an arbitrarily tumbling asymmetric vehicle about the maximum inertia axis. A first-order gradient optimization technique is used to minimize angular velocity components along the intermediate and minimum inertia axes. This method permits a wide range of initial guesses for mass position history. Motion of the control mass is along a linear track fixed in the vehicle. The control variable is taken as mass acceleration with respect to body coordinates. Motion is limited to defined quantities and a penalty function is used to insure a given range of positions. Numerical solutions of the optimization equations verify that minimum time detumbling is achieved with the largest permissible movable mass, length of linear track, and positions of the mass on the two coordinates perpendicular to the linear motion. Also, the mass should oscillate, about the zero point, on an axis parallel to the major principal axis. A minimum mass solution is obtained by fixing the time at the largest feasible value. The optimal method permits detumbling in about one-fourth the time when compared to a force control law formulation available in the literature. Since stabilization may require hours, this reduction in time is very significant.

In regard to minimum mass, the optimization permits the use of a much smaller mass for detumbling in the same time. This mass reduction is quite substantial since very large masses are required. Use of this control system for actual operations in space is feasible since the velocity and acceleration of the mass, and the power requirement, are low. It should be noted that the control technique utilizes an open loop solution in real time. In addition, the technique need not be restricted to attaining simple spin about the maximum inertia axis; geometric axes may be specified.

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NOMENCLATURE

$\vec{a}$ = Inertial acceleration of the origin of coordinates which is fixed at the center of mass of the main body

$\vec{f}_m$ = Force on the control mass

f_{m_x} = x component of $\vec{f}_m$

f = n column vector composed of f_i

f_i = Functions to which x_{s_i} are equal

$\vec{F}$ = Resultant of external forces

F_P = Function on which an inequality constraint is placed

Fp_{max} = Maximum permitted value of F_P

Fp_{fixed} = Actual Fp_{max} used in computation

$\vec{H}$ = Angular momentum of the system with respect to the origin of coordinates

$\vec{H}_m$ = Angular momentum of the movable control mass with respect to the origin of coordinates

$\vec{H}_B$ = Angular momentum of the main body with respect to the origin of coordinates

$H_{B_x}, H_{B_y}, H_{B_z}$ = Component of $\vec{H}_B$ along the x, y, and z axes, respectively

$\vec{H}_{cm}$ = Angular momentum of the system with respect to its own center of mass

$\vec{i}, \vec{j}, \vec{k}$ = Orthogonal unit vectors of coordinate frame fixed in the main body

I_x, I_y, I_z = Moments of inertia of the main body

I_{xy}, I_{xz}, I_{yz} = Products of inertia of the main body

$\bar{\bar{I}}$ = Inertia dyadic of the main body

I_{max} = Maximum moment of inertia of the system

NOMENCLATURE (Continued)

I_{min} = Minimum moment of inertia of the system

$I_{\psi\psi}$ = (q x q) matrix required in the first-order gradient method

$I_{\psi J}$ = q column vector required in the first-order gradient method

I_{JJ} = Scalar required in the first-order gradient method

J = Performance index to be minimized

K = Arbitrary constant associated with the penalty function technique

K_1, K_2 = K for P_1 and P_2, respectively

L = Integrand of performance index

m = Mass of movable object

M = Mass of main body

p = n column vector of influence functions

P = Auxiliary state variable due to penalty function

P_1, P_2 = P for the two inequality constraints on x_{max} and x_{min}, respectively

$\vec{r}$ = Position vector from center of mass of main body to the point mass

$\vec{r}_c$ = Position vector from center of mass of main body to the center of mass of the system

$\vec{R}_o$ = Position vector from inertial origin to the center of mass of the main body

$\vec{R}_c$ = Position vector from inertial origin to the center of mass of the system

R = (n x q) matrix of influence functions

NOMENCLATURE (Continued)

$\vec{S}$ = First moment of mass of the system

t = Time

t_0 = Initial time

t_f = Final time

T = Rotational kinetic energy of the system with respect to its own center of mass

u = m column vector of control variables

u_1 = Control variable equal to $\ddot{x}$

W = (m x m) position definite matrix required in the first-order gradient method

x,y,z = Coordinates corresponding to $\vec{i}$, $\vec{j}$, and $\vec{k}$, respectively

x_{max}, x_{min} = Maximum and minimum permitted mass positions on the x axis, respectively

x_h, x_l = Actual values used for x_{max} and x_{min}, respectively, during computation

x_s = n column vector of state variables

x_{s_i} = State variable

X,Y,Z = Coordinate axes fixed at the center of mass of the main body

β = State variable equal to $\dot{x}$

$\vec{\Gamma}$ = External moment

ε = Constant required in the first-order gradient method

μ = Equivalent mass equal to $mM/(M + m)$

ν = q column vector required in the first-order gradient method

NOMENCLATURE (Continued)

ψ = Terminal constraint

ψ_1, ψ_2 = ψ for P_1 and P_2, respectively

$\vec{\omega}$ = Angular velocity of spacecraft

$\omega_x, \omega_y, \omega_z$ = Components of $\vec{\omega}$ along $\vec{i}$, $\vec{j}$, and $\vec{k}$, respectively

$\omega_1, \omega_2, \omega_3$ = Components of $\vec{\omega}$ along the maximum, intermediate, and minimum inertia axes of the main body, respectively

$\omega_{2_{max}}, \omega_{3_{max}}$ = Maximum values of ω_2 and ω_3 desired

CHAPTER I

INTRODUCTION

General Introduction

Future manned spacecraft might be subjected to tumbling. Such a situation may result from collision with another vehicle, thruster malfunction, propellant tank rupture, or escaping atmosphere. For example, the Salyut 2 space station orbited by the Soviet Union on April 3, 1973 went into a tumbling mode which is believed to be a result of an explosion or a wildly firing thruster.[1] A study by Kaplan[2] on tumbling causes showed that a future manned space station configuration may be subjected to a tumbling state with angular velocities up to two RPM if collision occurs with a space shuttle. Escaping atmosphere will yield about the same state. Tank rupture may result in angular velocities of approximately one-half RPM. This tumbling should be immediately alleviated for crew safety and minimization of damage to the vehicle. Specifically, the crew would be subjected to oscillating accelerations. Hard docking by a rescue vehicle would be very difficult and would require a large fuel expenditure. In addition, it could not be implemented immediately. This time constraint also holds for other external detumbling methods such as fluid impingement. Hence, an internal control system is desirable.

Tumbling stabilization of space vehicles may be achieved by passive or active control devices. Passive systems are used as wobble dampers. They eliminate the wobbling motion of spacecraft by using mechanical or fluid dampers to dissipate energy until the minimum energy state is reached; their energy dissipation rates are low. This ultimately results in simple spin about the major principal axis. However, passive systems are designed for vehicles with high initial spin rates about the maximum inertia axis. Hence, they would not be appropriate for the future manned station mentioned above since it is normally in a non-spinning mode and a collision will result in the three angular velocity components of the vehicle being of the same order of magnitude. An active device such as a mass expulsion system may not be feasible since it requires long term, onboard storage of propellant. Momentum exchange systems probably would saturate in large corrective maneuvers and may require continuous operation. An active control device that does not have the restrictions mentioned above is an internal movable mass system. Movement of the mass will not affect the angular momentum vector, but it will affect the rotational kinetic energy. Therefore, internal mass motion can be used to decrease the rotational kinetic energy to a minimum which corresponds to the case of stable spin about the maximum inertia axis. For the case of the previously mentioned future manned space station, there will still be need of despinning; but, docking by a rescue vehicle with subsequent despinning would then be a relatively simple operation. Also, escape by crewmen, if necessary, will be easier from a spinning rather than a tumbling vehicle. Movable mass control systems, however, have been investigated for vehicles which fall within the assumption of symmetry, or small transverse angular velocities relative to the spin velocity, or both. Stabilization of a vehicle like the manned space station cited above which is not symmetric and which may experience angular velocity components of the same order of magnitude requires further investigation. A recent study of this general problem by Edwards[3] analyzed the rate of change of rotational kinetic energy in order to obtain stabilization; but, a long detumbling time and a large mass are required. Since it is important to have a mass as low as possible in space operations and, specifically for the distressed manned space station, to detumble as fast as possible, an optimal solution needs to be obtained.

Statement of the Problem

The objective of the research work presented here is to develop an optimal time and mass technique for obtaining the time history of internal control mass motion along a linear track in a tumbling space vehicle to achieve simple spin. No assumptions of symmetry or small transverse angular velocities relative to the spin velocity will be made. Specifically, the manned space station mentioned in the previous section will be used as the test case. A first-order gradient optimization technique will be used to obtain motions of the internal mass that will result in stabilization about the maximum inertia axis. A penalty function method will be used to limit the extreme positions of the movable mass. Effects on the solution due to changes of various parameters will also be investigated. These parameters, all referring to the movable mass, are: mass, length of the linear track, positions along the two coordinates perpendicular to the linear motion, position along the axis of mass motion of the center of the track, and the direction of the track. A study of these changes will yield guidelines for obtaining maximum effectiveness from a movable mass control system.

Summary of Work

The differential equations of motion for a spacecraft with one internal movable mass permitted to move along a linear track were written with respect to an arbitrary orthogonal coordinate system fixed at the center of mass of the main body, which is the spacecraft without the control mass. A first-order gradient method, minimizing the magnitudes of oscillations of angular velocity components along the intermediate and minimum inertia axes, was used to obtain control mass motions which yield simple spin about the maximum inertia axis. An IBM 370/165 computer was used to obtain the results. This quantitative analysis, along with a qualitative examination of the differential equations of motion, permitted evaluation of various parameters in order to determine values which will result in minimum time and mass detumbling to simple spin. For minimum time, the mass, length of the linear track, and positions of the mass on the two coordinates perpendicular to the linear motion should have magnitudes as large as possible. Also, the mass should oscillate, about the zero point, on an axis parallel to the maximum inertia axis. A minimum mass solution is obtained by fixing the time at the largest feasible value. Compared to Edwards, this optimal technique permits detumbling in about one-fourth the time. Since stabilization may require hours, this reduction in time is very significant. In regard to minimum mass, the optimization permits the use of a much smaller mass for detumbling in the same time. This mass reduction is quite substantial since very large masses are required.

CHAPTER II

PREVIOUS INVESTIGATIONS

Differential equations governing the angular motion of a space vehicle with moving internal parts have been obtained and discussed in several papers. Roberson[4] derives these equations relative to the composite center of mass of the system. Since the reference point is the composite center of mass, the inertia dyadic is a function of time. The equations permit relative translational and rotational motion within the spacecraft frame. Effects on the vehicle due to known motions of the parts are then examined. Grubin[5,6] also obtains and discusses differential equations governing the motion of a space vehicle with moving internal parts. However, his equations are referenced to the center of mass of the vehicle without moving parts. Therefore, the inertia dyadic of the main body is constant. He examines several two dimensional cases of mass translation in a vehicle; but his examples, like those of Roberson, deal with effects on the vehicle due to known motions of internal parts.

A number of papers in the literature deal with active control over the motion of internal parts in order to control the angular motion of a space vehicle. Since the differential equations are highly nonlinear, simplifying assumptions are made in order to determine solutions. The vehicle is assumed to have small angular velocity components about two of its axes relative to the angular velocity along the axis about which final steady spin is desired, or is assumed to be symmetric, or both.

Kane and Scher[7] analyzed the problem of active attitude control of a space vehicle with internal movable parts by considering its rotational kinetic energy. They noted that internal mass motion will not change the angular momentum vector since the net moment about the center of mass of the system is zero. However, internal mass motion will change the rotational kinetic energy of the system. Since the angular momentum vector is constant, rotational kinetic energy will be a maximum or a minimum if the rotation is about the minimum or maximum inertia axis of the vehicle, respectively. Thus, a tumbling space vehicle may be stabilized about the maximum or minimum inertia axis by internally moving a mass to either dissipate or add kinetic energy. If the space vehicle is symmetric, they further noted that the kinetic energy of the vehicle may be used as a guide to determine the motion of an internal mass in order to have simple spinning motion or a combination of precession and spin in which the angle between the inertially fixed angular momentum vector and the axis of symmetry takes on any preassigned value.

Specifically, the kinetic energy is written in terms of this angle and the maximum and minimum kinetic energies. For a specified angular momentum vector, the initial and final kinetic energies may now be written by specifying the initial and desired final angle. A trial and error procedure is then used in order to find the motion of an internal mass that will result in a change of the initial kinetic energy toward the desired final value. This procedure was applied to a solid uniform right-circular cylinder with a movable mass attached by a light rod.

Hopper[8] investigated the use of internal mass motion in order to decrease the precession angle of spacecraft spin stabilized about their minimum inertia axis. Energy dissipating mechanisms are excited during precessional motion of this type of vehicle. This causes an additional increase in the precession angle since spin about the minimum inertia axis is one of maximum kinetic energy. In order to overcome this effect and have spin about the minimum inertia axis, energy must be supplied to the system. Two active devices are presented: one is a rotary device and the other is a linear oscillator. Both are examined for use in an axisymmetric spacecraft. The rotary device consists of a mass attached to and able to rotate about the minimum inertia axis. By keeping the rotor at some constant offset angle relative to its position due to centrifugal force resulting from precession of the vehicle, positive work can be done and thereby increase the kinetic energy of the system. The linear device consists of a small mass undergoing forced oscillations along a linear path fixed within the vehicle and perpendicular to the spin axis. By oscillating the mass at the proper frequency and by proper control of phase, the driving motor will cause the kinetic energy of the spacecraft to increase.

Childs[9] investigated the problem of altitude stabilization of artificial-g space stations by a movable mass control system. The space station has the physical appearance of two rigid bodies connected by a long, slender tube and is spinning about its major principal axis. Movement by the crew may cause wobbling of the space station; that is, the angular momentum does not coincide with the maximum inertia axis. In order to damp this wobbling motion, the author placed a movable mass to the side of one of the two end pods. The mass was permitted to move inside of a tube which was parallel to the maximum inertia axis. By assuming small transverse angular velocities relative to the spin velocity, Childs was able to linearize the differential equations of motion. This permitted the formulation of a control law to govern the motion of the movable mass in order to damp the transverse angular rates. However, there was still some angular motion about one of the transverse axes after the application of the control law; the control system does not damp both transverse angular rates to zero.

Lorell and Lange[10] analyzed the use of internal moving masses to control the spin axis of a spinning satellite. Many satellites require control over the spin axis to an accuracy on the order of seconds of arc; two examples are communications satellites aiming high gain, narrow beam-width antennas and weather satellites scanning the surface of the earth for pictures. The control system has to take care of sensor-vehicle misalignments, motion of the principal axes of inertia, and body fixed disturbing torques. By assuming that the satellite is spinning about its axis of inertial symmetry, that the transverse angular rates are small relative to the spin velocity, and a specific geometry for four movable masses, the authors were able to simplify the differential equations of motion and use a linear control law.

As stated in the introduction, Edwards performed an independent investigation of the general problem of a vehicle with arbitrary angular velocity components and arbitrary principal moments of inertia concurrently with this thesis. He formulated a control law for the force on a movable mass, in the direction of motion of the mass, that will reduce the rotational kinetic energy of a tumbling vehicle and result in simple spin about the major principal axis. A formulation of a control law was made feasible by restricting the mass motion to lie along an axis parallel to the axis of maximum inertia and to pass through the zero point of that axis. The extreme mass positions permitted, on both sides of the zero position, are set in the control law; but, the formulated control law does not permit the mass to fully utilize the track available. Initially, the mass does go to an extreme position, but subsequent position peaks fall short of the extremes. Whether the extreme position will occur on the positive or negative side of the axis of mass motion is dependent on the initial conditions. Since these initial values are not known prior to installation of a control system on board a space vehicle, the tube for mass motion must extend on both sides of the zero position a distance equivalent to the extreme position permitted. The detumbling times and masses obtained by Edwards are very large. For the manned space station mentioned previously, over three hours are needed to obtain stabilization using a movable object whose mass is 0.5% of the space station mass. An optimal time and mass analysis of the general problem is needed in order to reduce the mass and time required.

CHAPTER III

ANALYTICAL STUDY

Purpose and Procedure

The purpose of this part of the investigation is to obtain the differential equations of angular motion for a space vehicle with a small movable internal mass and present an approach to obtaining mass motions that result in simple spin about the maximum inertia axis. Specifically, the rates of change of three orthogonal components of the angular velocity vector of the spacecraft will be expressed in terms of these angular velocities and the motion of the movable mass. For these equations, the center of mass of the main body will be used as the reference point for an arbitrary body fixed coordinate frame since this will allow the moments and products of inertia of the main body to be constant. Also, expressions will be derived for angular momentum, rotational kinetic energy, and rate of change of this energy. There will be no assumptions made of symmetry or of small angular velocities about two of the axes relative to a third; the space vehicle will be assumed to have three separate principal moments of inertia and an arbitrary angular velocity vector such that its three principal axis components may be of the same order of magnitude.

These equations are highly nonlinear and it is not obvious what the mass motion should be in order to detumble a space vehicle to simple spin about its maximum inertia axis. Whatever the motion should be, it must have reasonable values for mass displacement, velocity, and acceleration relative to the vehicle; that is, the mass should stay within the maximum dimensions of the space vehicle and should not have velocities and accelerations larger than can be supplied by a driving motor. Considering the above and further noting that a tumbling manned vehicle should be detumbled as quickly as possible, and possibly an unmanned vehicle from the standpoint of preserving structural integrity, the use of an optimization technique seems to be a feasible approach. Specifically, a first-order gradient optimization technique will be used since there are no previous solutions on which to base an initial guess of the control variable; this optimization technique does not require the initial guess to be close to the optimal values. Neighboring extremal and quasi-linearization methods require good initial estimates of various parameters. Also, a penalty function method will be used to limit the extreme positions of mass motion.

Equations of Motion

The angular momentum equation for a rigid body with an arbitrary origin of coordinates is[5,6]

$$\vec{\Gamma} = \dot{\vec{H}} + \vec{S} \times \vec{a} \tag{1}$$

where $\vec{\Gamma}$ is the external moment, $\vec{H}$ is the angular momentum of the system, $\vec{S}$ is the first moment of mass of the system, and $\vec{a}$ is the inertial acceleration of the origin of coordinates. The desired equations of motion for a spacecraft with one small movable mass may be obtained from Equation (1) by fixing the origin of coordinates at the center of mass of the main body, which is the spacecraft without the movable mass. The geometry of the system is shown in Figure 1 where x, y, z is a coordinate system, with $\vec{i}$, $\vec{j}$, $\vec{k}$ unit vectors, fixed in the main body, whose origin is at the center of mass of the main body. The angular momentum, $\vec{H}$, may be separated into two parts: the angular momentum of the main body relative to its own center of mass, $\vec{H}_B$, and the angular momentum of the movable control mass with respect to the center of mass of the main body, $\vec{H}_m$. The angular momentum $\vec{H}_B$ may be expressed as

$$\vec{H}_B = H_{B_x}\vec{i} + H_{B_y}\vec{j} + H_{B_z}\vec{k} \tag{2}$$

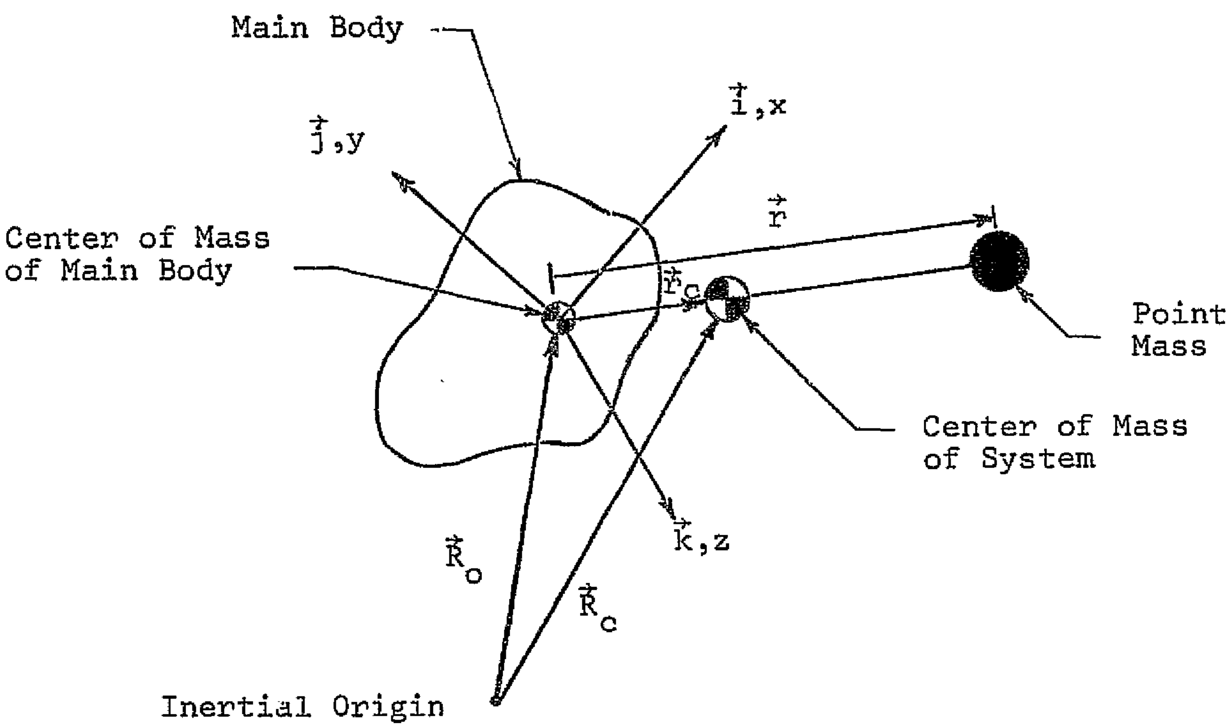

Figure 1. Geometry of System

where

$$H_{B_x} = I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z, \tag{3}$$

$$H_{B_y} = -I_{xy}\omega_x + I_y\omega_y - I_{yz}\omega_z, \tag{4}$$

$$H_{B_z} = -I_{xz}\omega_x - I_{yz}\omega_y + I_z\omega_z. \tag{5}$$

The inertial time derivative of $\vec{H}_B$ is

$$\dot{\vec{H}}_B = [\dot{\vec{H}}_B] + \vec{\omega} \times \vec{H}_B \tag{6}$$

where $[\dot{\vec{H}}_B]$ is the time derivative with respect to the body fixed x, y, z coordinate system of $\vec{H}_B$ which is given by Equations (2) through (5) and $\vec{\omega}$ is the angular velocity of the spacecraft which can be expressed as

$$\vec{\omega} = \omega_x\vec{i} + \omega_y\vec{j} + \omega_z\vec{k}. \tag{7}$$

The angular momentum $\vec{H}_m$ consists of the angular momentum of the mass about its own center of mass and the angular momentum of the mass moving about the center of mass of the main body:

$$\vec{H}_m = \vec{H}_{m_{\text{about its own center of mass}}} + m\vec{r} \times \dot{\vec{r}}. \tag{8}$$

Considering the movable mass, m, as a point mass, the first term on the right hand side of Equation (8) can be equated to zero. The inertial time derivative of $\vec{H}_m$ is then

$$\dot{\vec{H}}_m = m\vec{r} \times \ddot{\vec{r}}. \tag{9}$$

The acceleration of the mass with respect to the center of mass of the main body, $\ddot{\vec{r}}$, may be expressed as[11]

$$\ddot{\vec{r}} = [\ddot{\vec{r}}] + \vec{\omega} \times \vec{\omega} \times \vec{r} + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times [\dot{\vec{r}}] \tag{10}$$

where

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, \tag{11}$$

$$[\dot{\vec{r}}] = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}, \tag{12}$$

$$[\ddot{\vec{r}}] = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}, \tag{13}$$

and, with $\vec{\omega}$ given by Equation (7),

$$\dot{\vec{\omega}} = \dot{\omega}_x\vec{i} + \dot{\omega}_y\vec{j} + \dot{\omega}_z\vec{k}. \tag{14}$$

The first moment of mass of the system, $\vec{S}$, is

$$\vec{S} = m\vec{r}. \tag{15}$$

The main body does not contribute since the reference origin is its own center of mass. Using Figure 1, the acceleration of the origin, $\vec{a}$, may be expressed as

$$\vec{a} = \ddot{\vec{R}}_o = \ddot{\vec{R}}_c - \ddot{\vec{r}}_c \tag{16}$$

where

$$\ddot{\vec{R}}_c = \frac{\vec{F}}{M + m} \tag{17}$$

and, from the definition of center of mass,

$$\ddot{\vec{r}}_c = \frac{m\ddot{\vec{r}}}{M + m}. \tag{18}$$

$\vec{F}$ is the resultant of external forces and M is the mass of the main body. Considering the spacecraft to be in a circular orbit with zero jet thrusting, $\vec{F}$ may be set equal to zero. Using Equations (16) through (18), $\vec{a}$ can now be written as

$$\vec{a} = \ddot{\vec{R}}_o = -\frac{m}{M + m}\,\ddot{\vec{r}}. \tag{19}$$

Noting that

$$\dot{\vec{H}} = \dot{\vec{H}}_B + \dot{\vec{H}}_m \tag{20}$$

and placing Equations (9), (15), and (19) into Equation (1) gives

$$\vec{\Gamma} = \dot{\vec{H}}_B + \frac{mM}{M + m}\,\vec{r} \times \ddot{\vec{r}}. \tag{21}$$

For zero external moment and noting that the force on the control mass is

$$\vec{f}_m = m(\ddot{\vec{R}}_o + \ddot{\vec{r}}) \tag{22}$$

which, upon using Equation (19), becomes

$$\vec{f}_m = \mu\ddot{\vec{r}} \tag{23}$$

where

$$\mu = \frac{mM}{M + m}, \tag{24}$$

Equation (21) may be written as

$$\dot{\vec{H}}_B = -\vec{r} \times \vec{f}_m. \tag{25}$$

Thus, as Equation (25) shows, the force on the control mass may be considered as causing a moment to act on the main body. The mass will be permitted to move along a linear track parallel to the x axis. Placing Equations (2) through (7), (10) through (14), and (23) into Equation (25) yields three scalar equations which, when solved simultaneously, give the differential equations of motion in the desired form:

$$\dot{\omega}_x = \dot{\omega}_x(\omega_x, \omega_y, \omega_z, x, \dot{x}, \ddot{x}), \tag{26}$$

$$\dot{\omega}_y = \dot{\omega}_y(\omega_x, \omega_y, \omega_z, x, \dot{x}, \ddot{x}), \tag{27}$$

$$\dot{\omega}_z = \dot{\omega}_z(\omega_x, \omega_y, \omega_z, x, \dot{x}, \ddot{x}). \tag{28}$$

The full equations for $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$ are given in Appendix A. Since the motion is along an axis parallel to the x axis,

$$\dot{y} = \ddot{y} = \dot{z} = \ddot{z} = 0 \tag{29}$$

and the y, z positions of the mass may be arbitrarily fixed. Other constants that have to be specified are moments and products of inertia, and masses of the main body and the movable object. The control mass was permitted to have motion parallel to just one axis since then only one control variable will need to be specified; this fact will become important in the following section which deals with the optimization technique. However, the direction of the x axis, and, therefore, the direction of mass motion, relative to the spacecraft may be changed arbitrarily by appropriately changing the moments and products of inertia in the equations of motion given in Appendix A.

The total angular momentum vector with respect to the center of mass of the system, $\vec{H}_{cm}$, must remain constant during control mass motion. This can be seen by noting that[11]

$$\vec{\Gamma} = \dot{\vec{H}}_{cm}, \tag{30}$$

and, since there is no external moment during mass motion, $\vec{H}_{cm}$ is constant. An expression for this total angular momentum vector can be obtained by dividing it into two parts: that due to the rotation of the main body and that due to the motions of the centers of mass of the main body and the movable object about the center of mass of the system. The former is simply $\bar{\bar{I}} \cdot \vec{\omega}$ where $\bar{\bar{I}}$ is the inertia dyadic. The latter can be expressed as $\mu\vec{r} \times \dot{\vec{r}}$ by considering the two-body problem composed of the main body and the movable mass as an equivalent one-body problem which is the equivalent mass μ moving at a distance $\vec{r}$ from the center of mass of the system.[11] Thus, we have

$$\vec{H}_{cm} = \bar{\bar{I}} \cdot \vec{\omega} + \mu\vec{r} \times \dot{\vec{r}} \tag{31}$$

where

$$\dot{\vec{r}} = [\dot{\vec{r}}] + \vec{\omega} \times \vec{r}. \tag{32}$$

With this total angular momentum vector constant, the rotational kinetic energy, T, can assume the following values:

$$\frac{H_{cm}^2}{2I_{max}} \le T \le \frac{H_{cm}^2}{2I_{min}} \tag{33}$$

where I_{max} and I_{min} are the spacecraft's maximum and minimum moments of inertia. Specifically, using the same analysis as was used to obtain H_{cm}, we can write the following expression for kinetic energy relative to the center of mass of the system during mass motion:

$$T = \frac{1}{2}\vec{\omega} \cdot \bar{\bar{I}} \cdot \vec{\omega} + \frac{1}{2}\mu\dot{\vec{r}} \cdot \dot{\vec{r}}. \tag{34}$$

To have simple spin about the maximum inertia axis, the tumbling vehicle's rotational kinetic energy, which initially is some constant value, must be decreased to the value associated with this simple spin, $H_{cm}^2/2I_{max}$. It should be noted that simple spin about the major principal axis of the spacecraft can essentially be considered, in this investigation which uses a small mass, as simple spin about the major principal axis of the main body. If the control mass is large and far from the center of mass of the main body, the orientation relative to the spacecraft of the maximum inertia axis of the main body may be quite different from that of the spacecraft. In that case, the maximum inertia axis of the spacecraft should be used when referring to simple spin about the maximum inertia axis. Since the change in the kinetic energy of the system composed of a rigid main body and a movable mass requires work to be done and the only source of work is the force $\vec{f}_m$ acting on the control mass m which moves a distance $d\vec{r}$, we have

$$d(\text{Work}) = \vec{f}_m \cdot d\vec{r}. \tag{35}$$

This equation can also be obtained by considering $\vec{f}_m$ which is given by Equation (23) as acting on the equivalent mass m and causing a displacement $d\vec{r}$. For mass motion along a linear track parallel to the x axis, the right hand side of Equation (35) may be written as $f_{m_x}dx$. Therefore, we can write the following expression for the rate of change of kinetic energy which is equal to the rate at which work is being done:

$$\dot{T} = f_{m_x}\dot{x}. \tag{36}$$

It should be noted that $\dot{T}$ given by Equation (36) determines the power that will be required.

Optimal Control

A first-order gradient algorithm for the following problem is available in the literature:[12,13] with u and x_s defined as the column vectors of the control and state variables, respectively, $u_1(t)$, ... $u_m(t)$ must be found in order to minimize

$$J = \int_{t_o}^{t_f} L[x_{s_1}(t), \ldots x_{s_n}(t), u_1(t), \ldots u_m(t), t]dt \tag{37}$$

where

$$\dot{x}_{s_i} = f_i(x_{s_i}, \ldots x_{s_n}, u_1, \ldots u_m, t), i=1, \ldots n \tag{38}$$

with $x_s(t_0)$, t_0, t_f specified, and terminal equality constraints on q of the x_{s_i} variables, each represented by

$$\psi[x_{s_i}(t_f)] = x_{s_i}(t_f) - x_{s_{i_{final}}} = 0 \tag{39}$$

with the q desired terminal values, $x_{s_{i_{final}}}$, specified. Inequality constraints of the form

$$F_P(x_s, u, t) \leq F_{P_{max}} \tag{40}$$

may be handled by using a penalty function technique which converts inequality constraints to terminal constraints.[14] Specifically, an auxiliary state variable P is defined as

$$\dot{P} = \begin{cases} K[F_P(x_s, u, t) - F_{P_{fixed}}]^2, & F_P \geq F_{P_{fixed}} \\ 0, & F_P < F_{P_{fixed}} \end{cases} \tag{41}$$

and then the state variable P is forced to approach a value of zero by specifying the following form of Equation (39):

$$\psi = P(t_f) = 0. \tag{42}$$

In Equation (41), K is an arbitrary constant, and Fp_{fixed} is chosen to be smaller in magnitude than Fp_{max} since, in the first-order gradient method, P needs to exist in order to be controlled. If Fp_{fixed} is set equal to Fp_{max}, then some violation of the inequality constraint will have to be accepted.

The problem specified in the previous section will now be put into the form required for the application of the first-order gradient method. The differential equations of motion given by Equations (26) through (28) can be put into the form of Equation (38) by the following substitutions:

$$\dot{x} = \beta \tag{43}$$

and, since $\ddot{x}$ is equal to $\dot{\beta}$,

$$\dot{\beta} = u_1. \tag{44}$$

The state variables, x_{s_i} with i = 1, ... 5, are ω_x, ω_y, ω_z, x, and β, respectively. The one control variable is u_1; more control variables would be needed if the mass was not restricted to move parallel to one axis. The equations of motion can now be written as follows:

$$\dot{\omega}_x = \dot{\omega}_x(\omega_x, \omega_y, \omega_z, x, \beta, u_1), \tag{45}$$

$$\dot{\omega}_y = \dot{\omega}_y(\omega_x, \omega_y, \omega_z, x, \beta, u_1), \tag{46}$$

$$\dot{\omega}_z = \dot{\omega}_z(\omega_x, \omega_y, \omega_z, x, \beta, u_1), \tag{47}$$

$$\dot{x} = \beta, \tag{48}$$

$$\dot{\beta} = u_1. \tag{49}$$

In order to have simple spin about the maximum inertia axis, ω_2 and ω_3 which are angular velocity components along the intermediate and minimum inertia axes need to be minimized. Thus, modeling after regulator problems, the performance index J given by Equation (37) will be expressed as

$$J = \frac{1}{2} \int_{t_0}^{t_f} \left(\frac{\omega_2^2}{\omega_{2_{max}}^2} + \frac{\omega_3^2}{\omega_{3_{max}}^2}\right) dt \tag{50}$$

where $\omega 2_{max}$ and $\omega 3_{max}$ are the maximum magnitudes that are desired. Ideally, these values should be zero to have pure, simple spin about the major principal axis. However, in practice, these maximum values will be set at some very small magnitudes. The variables ω_2 and ω_3, of course, need to be expressed in terms of the state variables. If, for example the x, y, and z axes are aligned with the maximum, intermediate, and minimum inertia axes, respectively, then ω_1, ω_2, and ω_3 are equal to ω_x, ω_y, and ω_z. The x position of the control mass is limited as follows:

$$x_{min} \leq x \leq x_{max} \tag{51}$$

with the values x_{min} and x_{max} arbitrarily set. In order to apply the penalty function technique, Equation (51) will be expressed as

$$x \leq x_{max} \tag{52}$$

and

$$-x \leq -x_{min}. \tag{53}$$

Thus, there will be two auxiliary state variables, P_1 and P_2, associated with Equations (52) and (53), respectively; state variables x_{s_6} and x_{s_7} will refer to P_1 and P_2. Specifically, we have the following additional equations of motion:

$$\dot{P}_1 = \begin{cases} K_1(x - x_h)^2, & x \geq x_h \\ 0, & x < x_h \end{cases} \tag{54}$$

and

$$\dot{P}_2 = \begin{cases} K_2(-x + x_1)^2, & -x \geq -x_1 \\ 0, & -x < -x_1 \end{cases} \tag{55}$$

where x_h is less than x_{max} and x_1 is greater than x_{min}. Using Equation (42), we can now place terminal constraints on the two auxiliary state variables as follows:

$$\psi_1 = P_1(t_f) = 0 \tag{56}$$

and

$$\psi_2 = P_2(t_f) = 0. \tag{57}$$

The steps of the first-order gradient optimization technique for the problem just specified can be written as:[12,13]

1. Estimate $u_1(t)$ and integrate Equations (45) through (49), (54), and (55) forward with the known initial conditions $x_s(t_0)$. Store $x_s(t)$, $u_1(t)$, ψ_1, and ψ_2.

2. By backward integration of the following influence function equations determine and store the n column vector p(t) and the (n x q) matrix R(t) where n equals seven and q equals two:

$$\dot{p} = -\left(\frac{\partial f}{\partial x_s}\right)^T p - \left(\frac{\partial L}{\partial x_s}\right)^T \tag{58}$$

and

$$\dot{R} = -\left(\frac{\partial f}{\partial x_s}\right)^T R, \tag{59}$$

with

$$p(t_f) = 0 \tag{60}$$

and

$$R_{ij}(t_f) = \begin{cases} 0, & i = 1, \ldots 5 \text{ and } j = 1, 2 \\ 1, & i = 6 \text{ and } j = 1 \\ 0, & i = 6 \text{ and } j = 2 \\ 0, & i = 7 \text{ and } j = 1 \\ 1, & i = 7 \text{ and } j = 2 \end{cases} \tag{61}$$

where i = 6 and 7 correspond to the auxiliary state variables P_1 and P_2, respectively.

3. Evaluate the following integrals:

$$I_{\psi\psi} = \int_{t_0}^{t_f} R^T \frac{\partial f}{\partial u_1} W^{-1} \left(\frac{\partial f}{\partial u_1}\right)^T R \, dt$$

$(q \times q)$ matrix, (62)

$$I_{\psi J}^T = \int_{t_0}^{t_f} \left(p^T \frac{\partial f}{\partial u_1} + \frac{\partial L}{\partial u_1}\right) W^{-1} \left(\frac{\partial f}{\partial u_1}\right)^T R \, dt$$

q row vector, (63)

$$I_{JJ} = \int_{t_0}^{t_f} (p^T \frac{\partial f}{\partial u_1} + \frac{\partial L}{\partial u_1}) W^{-1} [(\frac{\partial f}{\partial u_1})^T p + (\frac{\partial L}{\partial u_1})^T] dt$$

scalar, (64)

where W is just a (1 x 1) positive-definite matrix since there is only one control variable.

4. Select a $\delta\psi$, given by

$$\delta\psi = \begin{bmatrix} \delta\psi_1 \\ \delta\psi_2 \end{bmatrix}, \tag{65}$$

which will bring ψ_1 and ψ_2, given by Equations (56) and (57), closer to zero on the next iteration. Specifically, choose

$$\delta\psi_1 = \varepsilon P_1(t_f) \tag{66}$$

and

$$\delta\psi_2 = \varepsilon P_2(t_f) \tag{67}$$

where

$$0 < \varepsilon \leq 1. \tag{68}$$

Then, determine the q column vector ν from

$$\nu = -[I_{\psi\psi}]^{-1}(\delta\psi + I_{\psi J}). \tag{69}$$

5. Repeat steps 1 through 4 with the following improved $u_1(t)$:

$$u_1(t)_{new} = u_1(t)_{old} + \delta u_1(t) \tag{70}$$

where

$$\delta u_1(t) = -[W(t)]^{-1}\{\frac{\partial L}{\partial u_1} + [p(t) + R(t)\nu \frac{\partial f}{\partial u_1}]\}^T. \tag{71}$$

Terminate when ψ_1, ψ_2, and $I_{JJ} - I_{\psi J}^T I_{\psi\psi}^{-1} I_{\psi J}$ equal zero to the desired degree of accuracy. These steps have been speciliazed to the problem consisting of seven state variables, with terminal constraints on the sixth and seventh state variables, and one control variable.

Numerical Solution

The five steps of the first-order gradient optimization technique were programmed on an IBM 370/165 computer. The computer program, given in Appendix B, consists of a main program with three subroutines. In the main program, the necessary variables are specified. The variables needed for the specification of a case which is to be studied are listed in the beginning of Appendix B; their computer language names are also specified. The subroutines carry out the integrations and changes in the control variable as prescribed in the five steps. The differential equations are solved by using a fourth-order Runge-Kutta algorithm. Integrals are handled by using a library program consisting of an extended five-point Newton-Cotes quadrature formula.

Implementation and Nature of the Optimal Control

The optimal solution obtained is really a local optimal. Standard numerical optimization techniques like the first-order gradient do not necessarily yield the absolute minimum. Initial guess of the solution will determine which local optimal is obtained; this optimal may be the absolute optimal. Of course, for a specific problem there may be only one minimum and, therefore, the solution is the absolute minimum. An examination of numerical results and a comparison to non-optimal solutions will give an indication of the nature of the solution obtained. Results of this investigation are presented and compared to those of Edwards in the next chapter.

The optimal control may be implemented quite easily. The first-order gradient method yields a time history of the mass motion. Thus, the control system need just monitor and change the position of the mass. The position may be monitored by a simple mechanical device. It should be noted that any means may be used to move the mass. In actual space operations, three orthogonal angular velocities of a tumbling vehicle will be sensed by rate gyros and extrapolated to a time a few minutes in the future using Euler moment equations. These future angular velocities will then be used as the initial conditions for the optimization equations. The optimal control will be numerically obtained and initiated at the chosen future time.

CHAPTER IV

RESULTS

Several parameters need to be specified before computer simulations may be run for minimum time stabilization. As will become evident, selection of some of these parameters will depend upon the specific satellite to which the movable mass control system is applied; that is, the dimensions of the satellite and the amount of time that can be permitted for detumbling. The choices for minimum time detumbling of the remaining parameters will also become evident; but, these parameters will not be dependent on the type of satellite to be controlled. Specifically, referring to the differential equations of motion in Appendix A and noting that the mass moves along an arbitrary x direction, these parameters are as follows: mass of the movable object, length of the linear track, y and z positions of the mass, point about which the mass oscillates, and direction of the x axis relative to the spacecraft. By examining Equation (25) and thinking in terms of moments applied about each axis, a qualitative preliminary analysis may be made as to the effect of these parameters on the time needed to detumble. Increasing the mass of the object will increase the force; thereby increasing the moment and permitting a decrease in the detumbling time. Increasing the length of the linear track or the y and z magnitudes will increase the moment arm, which should tend to increase the moment and result in a lower minimum time. It should be noted that changes in these parameters will also effect the force; but, the overall effect on the moment will probably be due to the change in the moment arm mentioned previously since the force consists of the relative difference of various terms. By increasing the x value of the point about which the mass oscillates, the moment arm is again increased. However, by permitting the mass to move further on one side of the zero x position than on the other, there may arise difficulties due to a larger moment in one direction than in the opposite. Changing the direction of the x axis relative to the spacecraft will affect the moment arms of the force components producing moments about the intermediate and minimum inertia axes. If the x axis is parallel with the maximum inertia axis, maximum control over the moment arms of the moments about the intermediate and minimum inertia axes will be available; this can be seen by noting that now the x axis is perpendicular to the intermediate and minimum inertia axes. This would seem to indicate that the orientation of the linear track should be parallel to the final spin axis, which is the major principal axis. However, in this case as in the other cases that involved changes in the moment arm, there are also changes in the force itself which are difficult to specify. The above qualitative analysis is not sufficient in itself to arrive at minimum time values for all the parameters being discussed. A quantitative analysis must be made.

The movable mass control system will be applied to a manned space station configuration which NASA is considering for the 1980's. This configuration is shown in Figure 2 with pertinent data being given in Table 1. The optimization procedure will permit the fastest detumbling possible with the movable mass. Based on the previous qualitative discussion of various parameters in the differential equations of motion, we will initially fix the parameters to yield the best detumbling sequence; further cases will vary these parameters in order to quantitatively determine the minimum time solution. The mass of the movable object will be set at 499 kg, which is 0.5% the mass of the manned space station. It will be permitted to travel approximately ±3.7 m about the zero position on the x axis. This axis of motion will be parallel to and have the same sense as the maximum inertia axis. For convenience, the y and z axes will be chosen to be parallel to and have the same sense as the intermediate and minimum inertia axes, respectively. Choosing the y and z positions are large as possible within the limits of the space station, we have 5.55 m

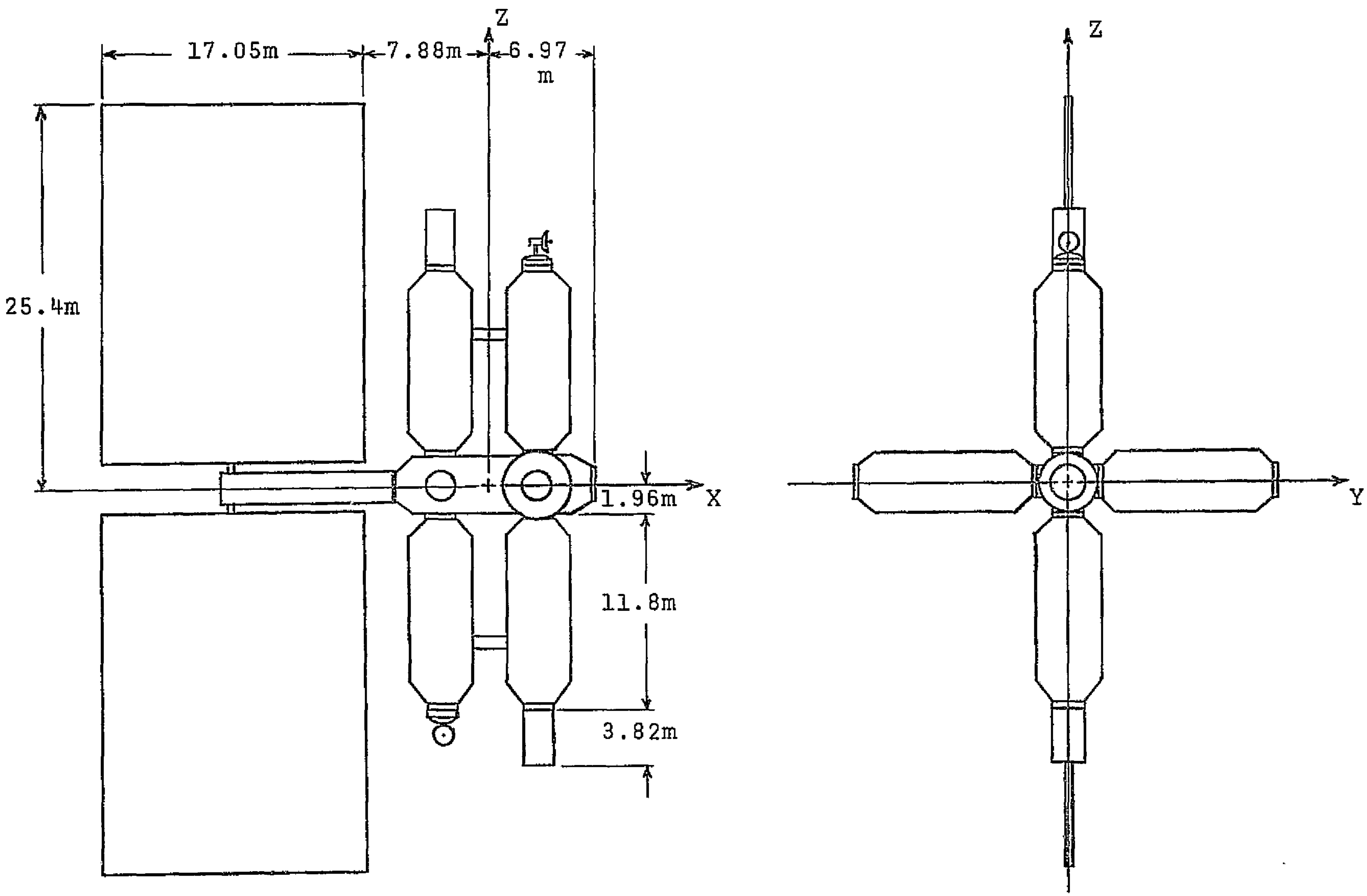

Figure 2. Manned Space Station[15]

Table 1. Manned Space Station Data[2,15]

Mass	99792 kg
Maximum Moment of Inertia	[illegible].74 x 10^6 kg-m^2
Intermediate Moment of Inertia	6.28 x 10^6 kg-m^2
Minimum Moment of Inertia	5.15 x 10^6 kg-m^2

Transformation matrix, body fixed X, Y, Z to principal 1, 2, 3, where 1, 2, 3 are the maximum, intermediate, and minimum moments of inertia axes respectively

$$\begin{bmatrix} 0.458 & -0.889 & 0.00676 \\ 0.889 & 0.458 & -0.00204 \\ -0.00128 & 0.00695 & 1.0 \end{bmatrix}$$

and -13.7 m. The initial angular velocity components along the 1-axis of maximum inertia, the 2-axis of intermediate inertia, and the 3-axis of minimum inertia will be chosen as 0.103 rad/sec, -0.199 rad/sec, and 0.000286 rad/sec. These values are based on a worst case tumbling analysis.[2] They represent the highest tumbling mode of the manned space station and are due to a collision between it and a space shuttle vehicle. If uncontrolled, the manned space station with a fixed 499 kg internal control mass would continue to tumble with ω_1 oscillating between 0.103 rad/sec and 0.192 rad/sec, ω_2 oscillating between -0.199 rad/sec and 0.199 rad/sec, and ω_3 oscillating between -0.118 rad/sec and 0.118 rad/sec. Flexibility effects, of course, will tend to decrease the rotational kinetic energy of the vehicle and alter the envelopes of oscillation. However, in the time periods involved in control, these limits of oscillation can be used as a reference for zero control mass motion. The effects of control by an internal movable mass system on the oscillations of ω_1, ω_2, and ω_3 are shown in Figure 3. The curves in this figure are the envelopes of oscillations of the principal axes angular velocity components. At 2,845 sec, the limits, for the penalty function, of mass motion along the x axis were set at $\pm 10^{-9}$ m in order to zero out the mass position, velocity, and

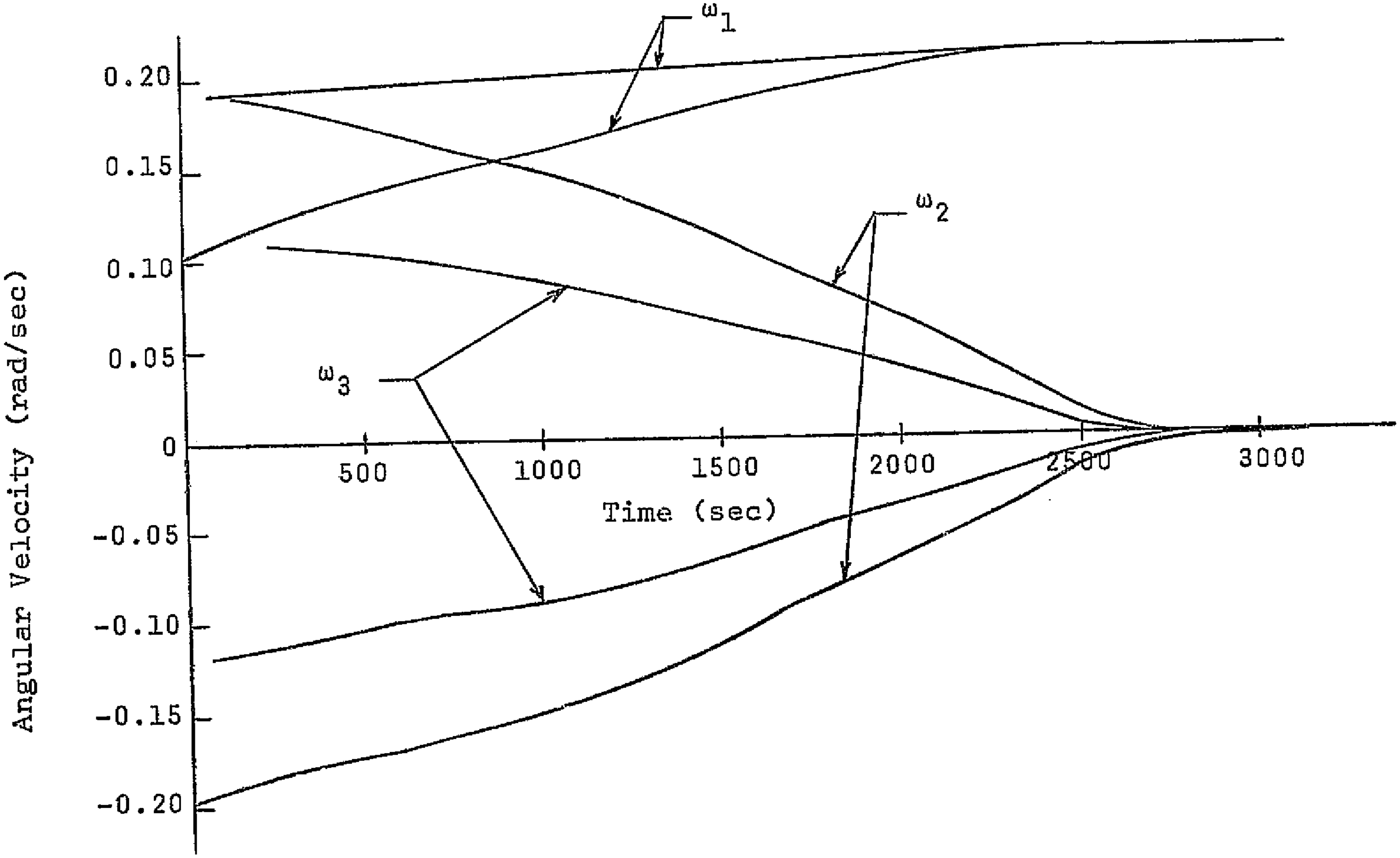

Figure 3. Envelopes of Angular Velocity Oscillations Using 0.5% Mass

acceleration. By 2,893 sec these three variables were essentially zeroed out, having values of 0.0466 m, -0.00322 m/sec, and 0.00141 m/sec^2. After this time, the mass was kept fixed at the zero x position; ω_1 remained at 0.212 rad/sec, ω_2 oscillated between -0.00152 rad/sec and 0.00142 rad/sec, and ω_3 oscillated between -0.000837 rad/sec and 0.000909 rad/sec. It should be noted that the percentage change from peak to peak of ω_2 and ω_3 is greater toward the end of the control time since the absolute decrease remains fairly constant. After about 3,000 sec, then, the peak values of ω_2 and ω_3 were reduced by more than 99% of their initial value. Of course, the mass control system could have been left on to reduce the ω_2 and ω_3 oscillations even further. For the manned space station discussed in this investigation, these values are sufficient since the effect felt by the crewmen is essentially that of simple spin and docking by a rescue and despinning vehicle can be made as if with a simple spinning body. Figure 4 shows the motion of the internal movable mass which results in detumbling of the manned space station. The mass position, x, at no time exceeds 3.7 m; after 2,893 sec, it is essentially equal to zero. The mass oscillates between its maximum permitted limits, utilizing the full linear track available to it. The velocity of

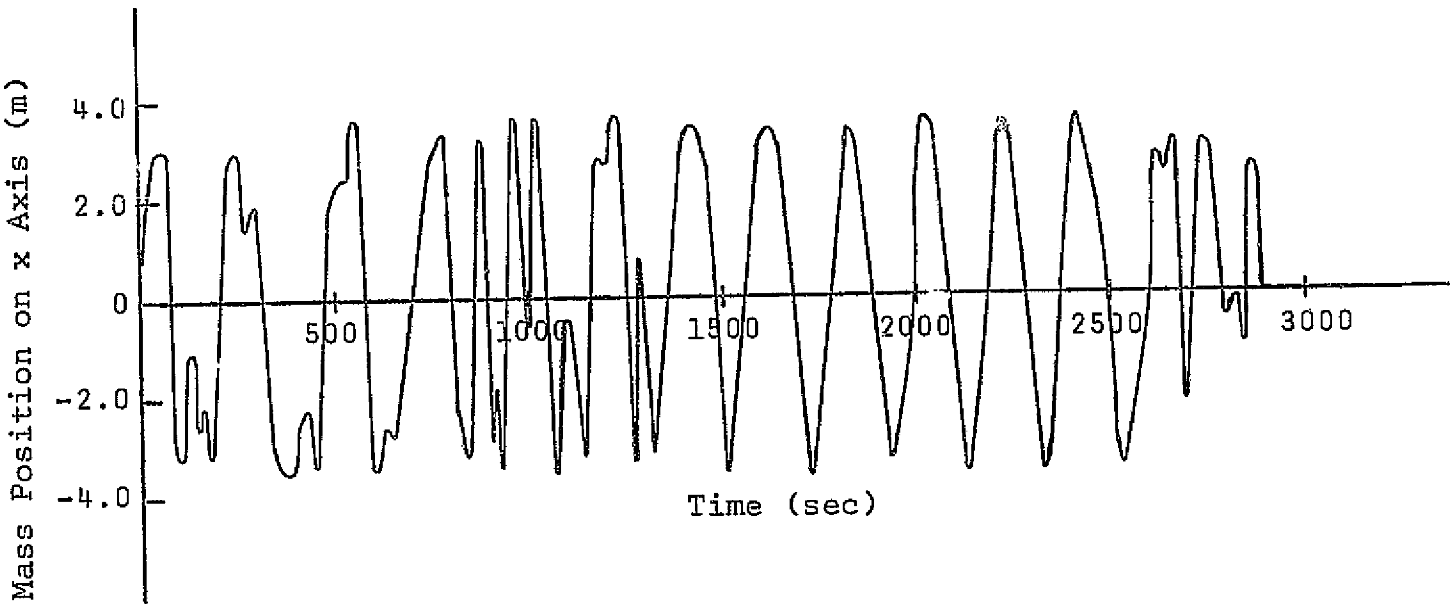

Figure 4. Motion of 0.5% Control Mass

the mass, $\dot{x}$, oscillates between -0.647 m/sec and 0.654 m/sec. The greatest mass acceleration, $\ddot{x}$, occurs during the zeroing out of the mass position, velocity, and acceleration; in this period it reaches values of -0.579 m/sec^2 and 0.465 m/sec^2. Also during the zeroing out, the force in the x direction, f_{m_x}, acting on the mass reaches its largest magnitude, 283 N. These values for mass velocity and acceleration, and for force, are reasonable. It should further be noted that the mass velocity and force maximum magnitudes occur during energy dissipation, $-\dot{T}$. During energy dissipation, the force in the x direction and the mass velocity are opposite in direction; here the control system is actually restraining the mass. As Kane and Scher[7] have pointed out, this energy dissipation could be used to provide useful power for the vehicle's systems. Energy has to be provided by the mass only when $\dot{T}$ is positive. Figure 5 shows the variation of $\dot{T}$, power with time. The positive power is much less than the negative. Much more energy is dissipated than added. Also, as this figure shows, the total energy that is to be supplied is quite reasonable. The maximum positive power is 48.4 watt, also a reasonable value; this value could have been reduced to a value similar to the other peaks by decreasing the simulation time increment. Time increments will be discussed later. The -300 watt value at 2,848 sec, during zeroing out, corresponds to energy

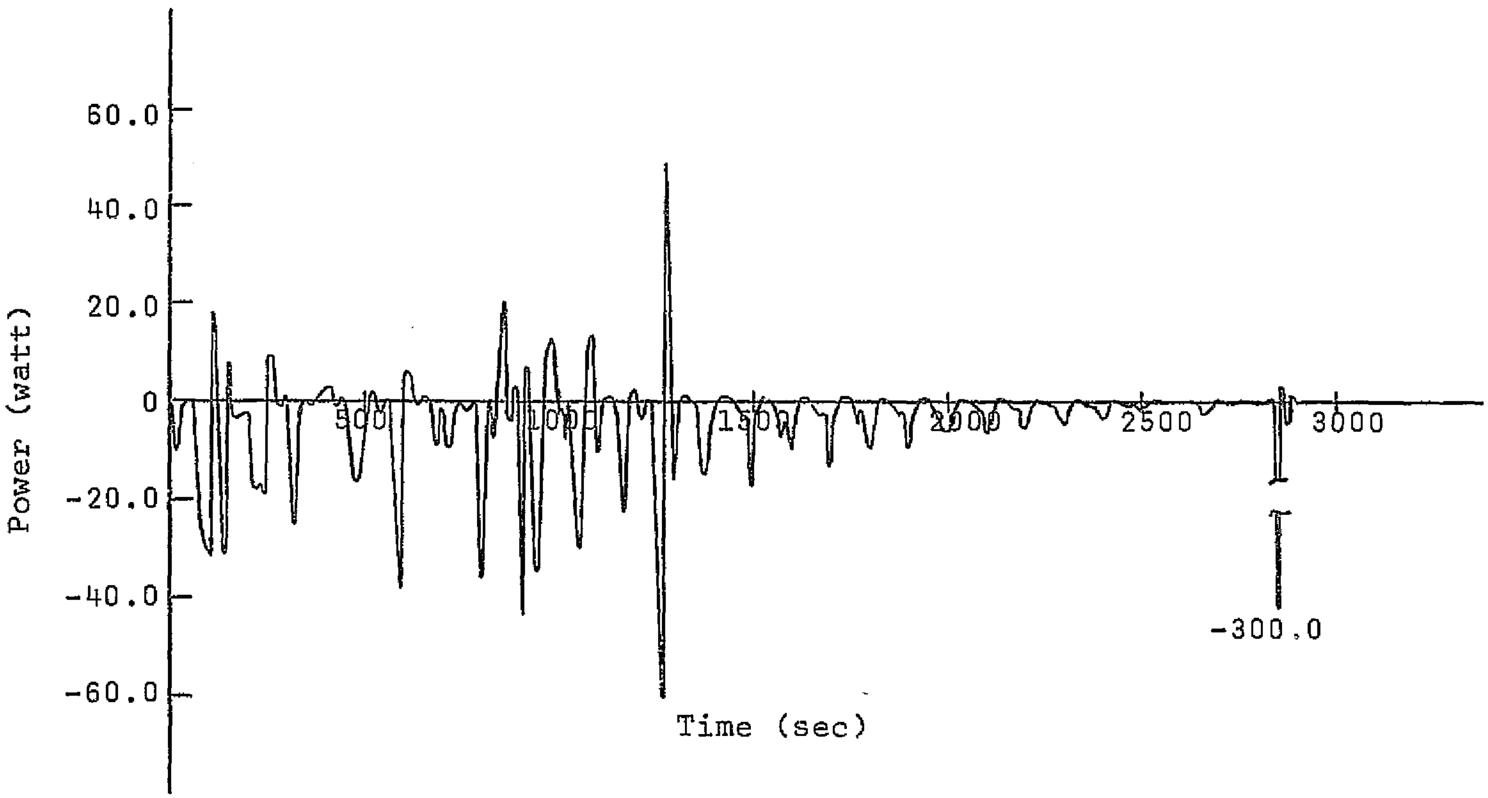

Figure 5. Power Required for 0.5% Mass Motion

dissipation; the mass is being restrained and the control system does not have to supply that energy. Figure 6 shows the decrease of rotational kinetic energy from its initial, before control mass motion, value of 1.62×10^5 joule to the final value for stable simple spin of 1.5×10^5 joule. At various points in this figure, the kinetic energy increases slightly and then resumes its downward curve. These increases, of course, correspond to energy addition to the movable mass control system, the positive power points on Figure 5. Furthermore, by superimposing Figure 4 for mass x position with Figure 5 for power during mass motion, it is seen that these energy addition points correspond to the points at which the mass direction of motion needs to be reversed in order not to exceed the extreme limits of motion that were previously set. Throughout the period of control mass motion, the angular momentum of the system relative to its center of mass remains fixed at the value it had before control mass motion was initiated, 1.45×10^6 kg-m^2/sec. The angular momentum vector remains constant since there are no external moments on the space vehicle. Therefore, as is evident, an optimal movable internal mass control system can be used to reduce the arbitrary tumbling of a general space vehicle to simple stable spin about the maximum inertia axis.

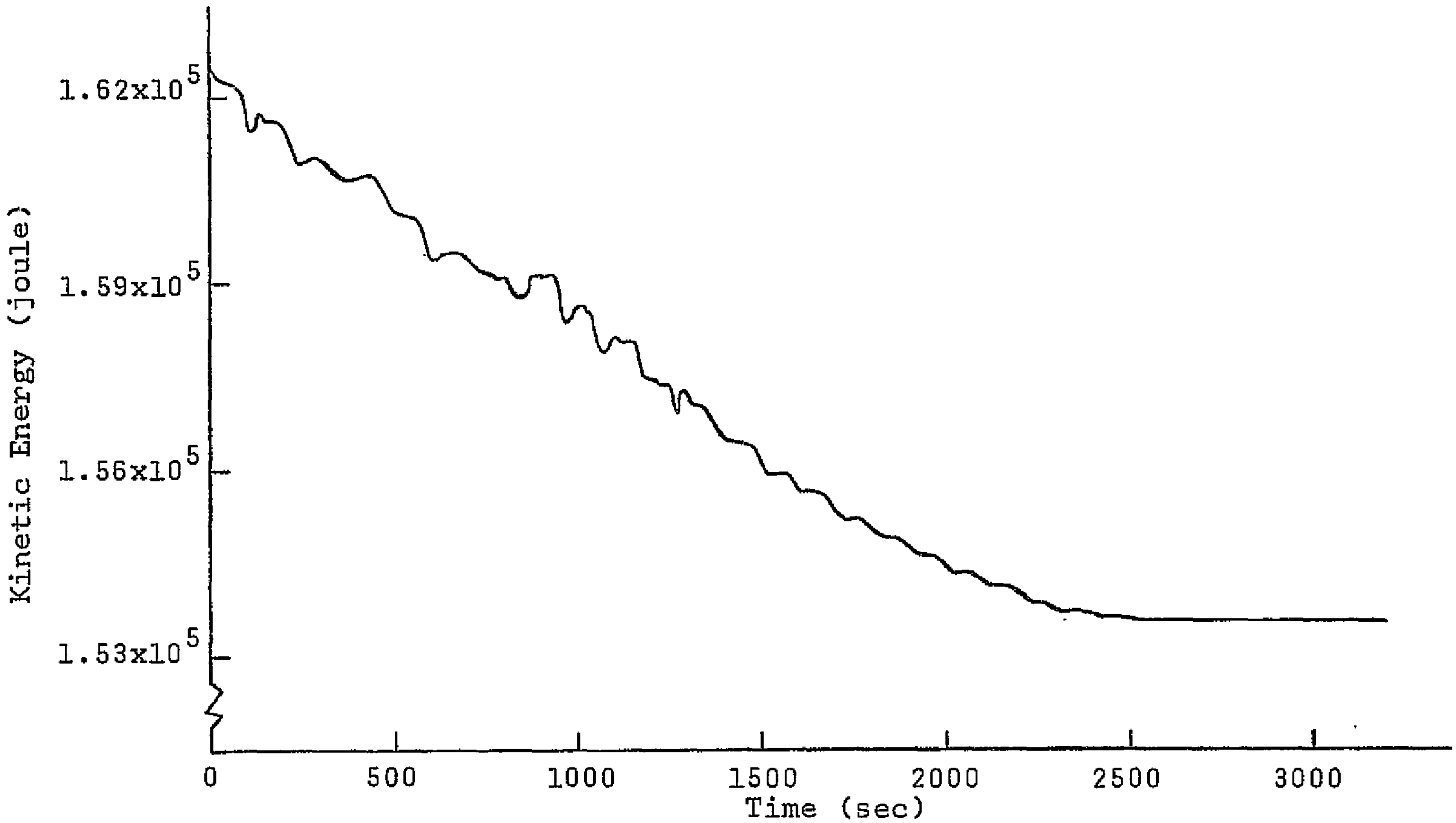

Figure 6. Decrease of Rotational Kinetic Energy Using 0.5% Mass

One valid comparison between the optimal control mass motion described in this investigation and the limited analysis of the force control law formulation would be to keep all parameters and initial conditions mentioned in the previous paragraph the same, including the permitted extreme positions on both the positive and negative sides of the mass motion axis; as stated in the previous investigation, the linear track required by the force control law method must extend, in the positive and negative directions, a distance equivalent to the maximum movement permitted. Doing this, the force control law yielded a decrease of the ω_2 envelope of oscillation to a magnitude of 0.00206 rad/sec at 11,050 sec, and the ω_3 envelope to a magnitude of 0.00124 at 11,005 sec. As was mentioned in the previous paragraph, the optimal analysis of this investigation decreased the magnitudes of the ω_2 and ω_3 envelopes to 0.00152 rad/sec and 0.000909 rad/sec respectively by about 2,900 sec. Thus, the optimal analysis permits detumbling in approximately one-fourth of the time required by the force control law analysis. Since the one-fourth value means that the crewmen will be subjected to a tumbling state for less than an hour compared to over three hours, it is quite significant. The force control analysis required only about one watt of peak positive power. As was shown, the power for the optimal control is greater, but still within the limits of production in spacecraft. Another comparison between the optimal analysis and the formulated force control law approach can be made by restricting the mass motion extreme positions in the former method to the actual extreme positions of the latter, keeping all other initial conditions and parameters the same; the values used in the case mentioned in the previous paragraph will again be used. The first 160 sec after commencing control mass motion were investigated. In this time interval, the force control law required mass position peaking of 0.631 m at 35 sec and -3.75 m at 155 sec; ω_2 and ω_3 peak at 0.1962 rad/sec and 0.1179 rad/sec. The next peaking of mass position occurs at 295 sec with a value of 2.23 m. The optimal mass control system was started at the 50 sec point of the force control sequence since, by that time, the mass position had peaked at only 0.631 m. Using the values at the 50 sec point, the optimal method peaked the mass position first at the largest positive limit and then at the negative one. This motion resulted in an ω_2 peak of 0.1947 rad/sec and an ω_3 peak of 0.1171 rad/sec. Without any control mass motion, the vehicle would have experienced an ω_2 peak of 0.1991 rad/sec and an ω_3 peak of 0.1182 rad/sec.

Thus, the optimal technique yielded about a one and one-half times greater decrease in the ω_2 value and slightly less than four times greater decrease in the ω_3 value. The actual effect on the time to detumble will be greater since the angular velocities are lower for the start of their next cycles and their periods are decreasing more; this is in comparison to the values yielded by the force control law method. Therefore, the optimal control mass motion technique, in addition to not having the restriction on the direction of mass motion nor on the point about which the mass oscillates, yields simple spin in a considerably faster time than the force control law method.

The effect of a change in the mass of the movable control object was investigated by using the case initially studied and only changing one parameter, the mass, from 499 kg to 998 kg; this new value for the mass is one percent that of the manned space station. Specifically, the control mass will again be permitted to move 3.7 m about the zero position on the x axis which is parallel to and has the same sense as the maximum inertia axis, with the y and z positions fixed at 5.55 m and -13.7 m, respectively. Also, the y and z axes are parallel to and have the same sense as the intermediate and minimum inertia axes.

The initial values for ω_1, ω_2, and ω_3 remain set at 0.103 rad/sec, -0.199 rad/sec, and 0.000286 rad/sec. Figure 7 shows the envelopes of oscillation of the principal axis angular velocity components. Comparing with Figure 3, it is seen that doubling the mass to one percent of the space station mass caused about a one-half decrease in the time to detumble to simple spin about the maximum inertia axis. After 1,485 sec, ω_2 is oscillating between -0.00186 rad/sec and 0.00199 rad/sec, and ω_3 between -0.00126 rad/sec and 0.000425 rad/sec. To reach similar oscillation ranges, the 0.5% mass required about 3,000 sec. Comparing this one percent case to Edwards, it is again evident that only about one-fourth the time is required; his force control law reduced ω_2 and ω_3 to peak magnitudes of 0.00292 rad/sec and 0.0032 rad/sec at 5,680 sec and 5,450 sec, with mass set at one percent.

The effect of a change in the length of the linear track is evident in every computer run. At first, the peaks of the ω_2 and ω_3 oscillations are lowered in magnitude considerably by having the mass move far out on the x axis. Each successive iteration improves on the control variables in order to decrease the extreme positions of the mass to prescribed limits. As the extreme limits are decreased, the peaks of the ω_2 and ω_3 oscillations increase

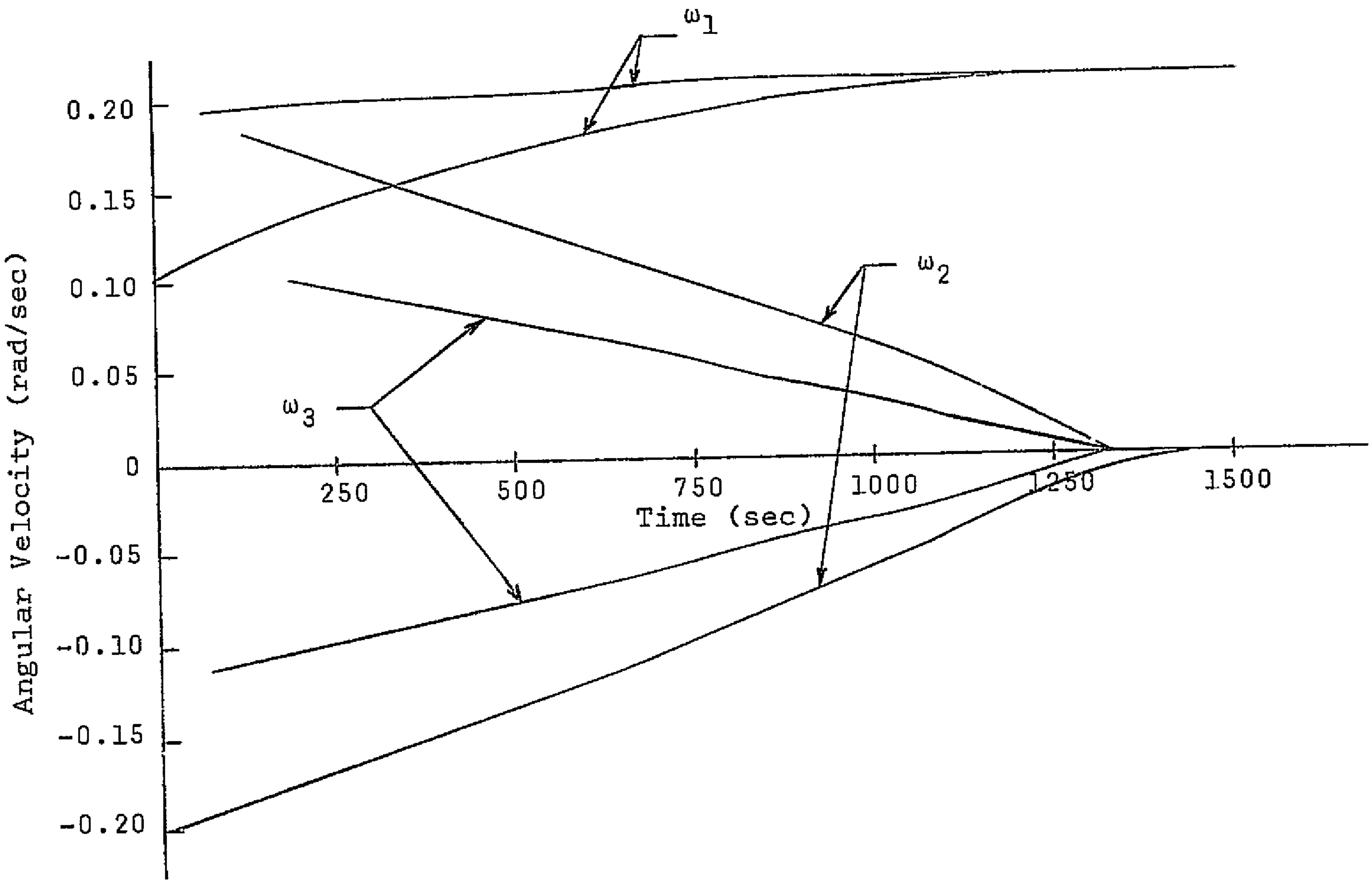

Figure 7. Envelopes of Angular Velocity Oscillations Using One Percent Mass

in magnitude. The above examined case of a one percent control mass will be used to show this effect. The first 100 sec of simulation are shown in Figure 8. After a few iterations, the mass position extremes are reduced to ±10 m. The resulting ω_1, ω_2, and ω_3 oscillations are plotted. After additional iterations, the mass position limits are -2.8 m and 3.4 m, and it is seen in this figure that the ω_2 and ω_3 oscillations have become larger in amplitude. Comparison to the no mass motion curves shows that the effect of extreme mass position change is quite substantial. It should be noted that, as the extreme limits of mass motion are decreased and cause an increase in the amplitudes of the ω_2 and ω_3 oscillations, an increase in total time to detumble will occur.

The changes in the oscillations of ω_2 and ω_3 due to a one-half decrease in the y and z positions of the control mass are examined by using the one percent mass case mentioned above and appropriately changing y and z. A 998 kg mass was permitted to move 3.7 m about the zero position on the x axis which is parallel to and has the same sense as the maximum inertia axis, with y and z fixed at 5.55/2 m and -13.7/2 m; the y and z axes are parallel to and have the same sense as the intermediate and minimum inertia axes. The only differences between the case that

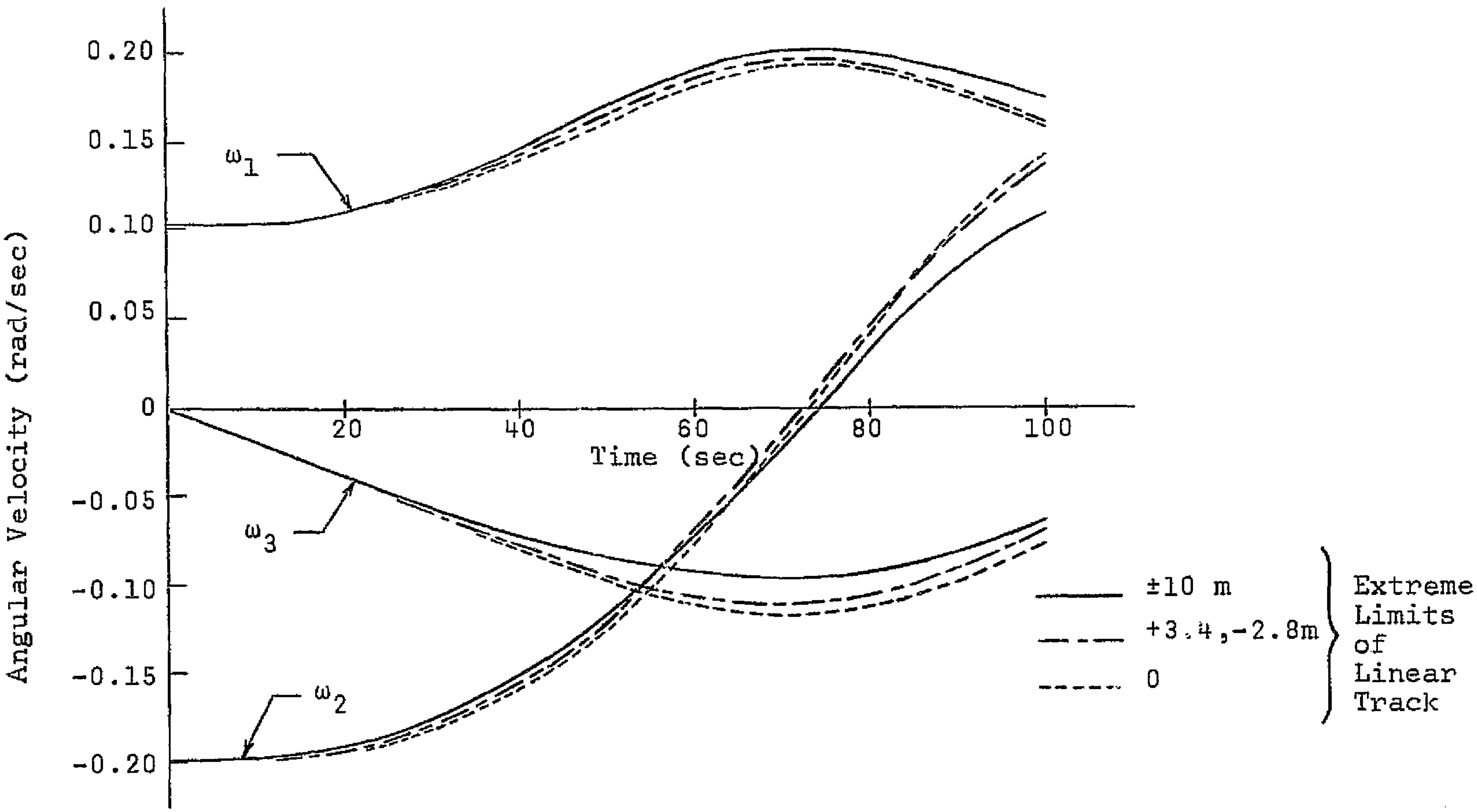

Figure 8. Oscillations of Angular Velocity Using One Percent Mass with Various Lengths for Linear Track

will be discussed now and the initially discussed 0.5% case are an increase in the mass of the movable object to one percent of the manned space station and a one-half decrease in the y and z positions which were 5.55 m and -13.7 m. Figures 9, 10, and 11 show the ω_1, ω_2, and ω_3 oscillations. For comparison, the ω_2 and ω_3 oscillations from the previous one percent case, which had the same parameters except for full y and z values of 5.55 m and -13.7 m, are shown. The limits of the ω_2 and ω_3 oscillations for no motion of a one percent mass are also noted. By comparing the various curves, it is seen that there is a definite increase in the magnitudes of the ω_2 and ω_3 oscillation peaks caused by lowering the y and z mass position magnitudes. The long term effect will be an increase in total detumble time due to the lower y and z magnitudes.

The effect of a control mass oscillating about a non zero position is shown in Figure 12. The initial one percent mass case was again used as a basis and the only parameter change was letting the mass oscillate about +10 m instead of the zero x position. Specifically, a 998 kg mass was permitted to move 3.7 m about the 10 m position on the x axis which is parallel to and has the same sense as the maximum inertia axis, with y and z fixed at 5.55 m and -13.7 m; the y and z axes are parallel

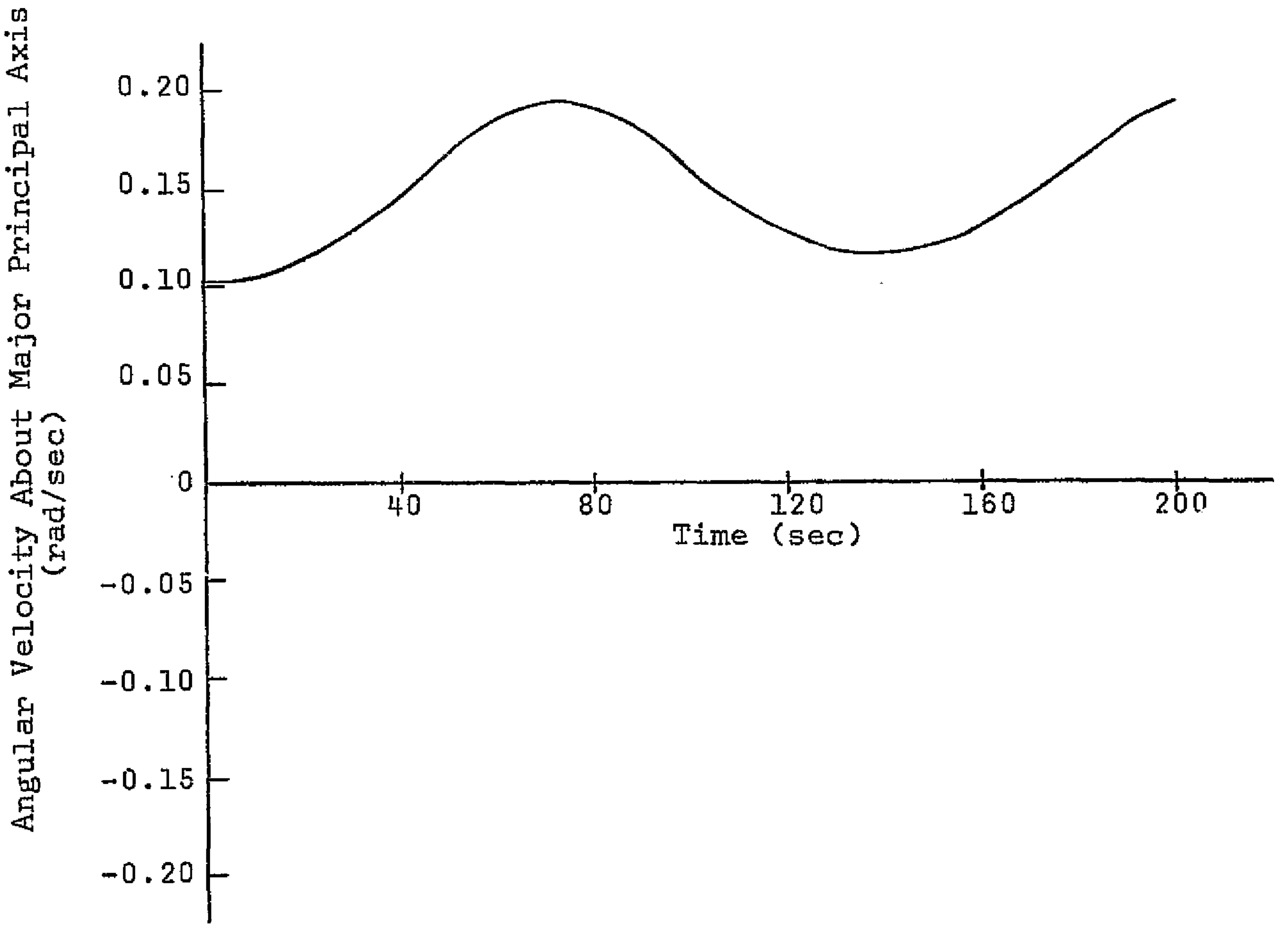

Figure 9. ω_1 Oscillation Using One Percent Mass with y and z Values of 5.55/2 m and -13.7/2 m

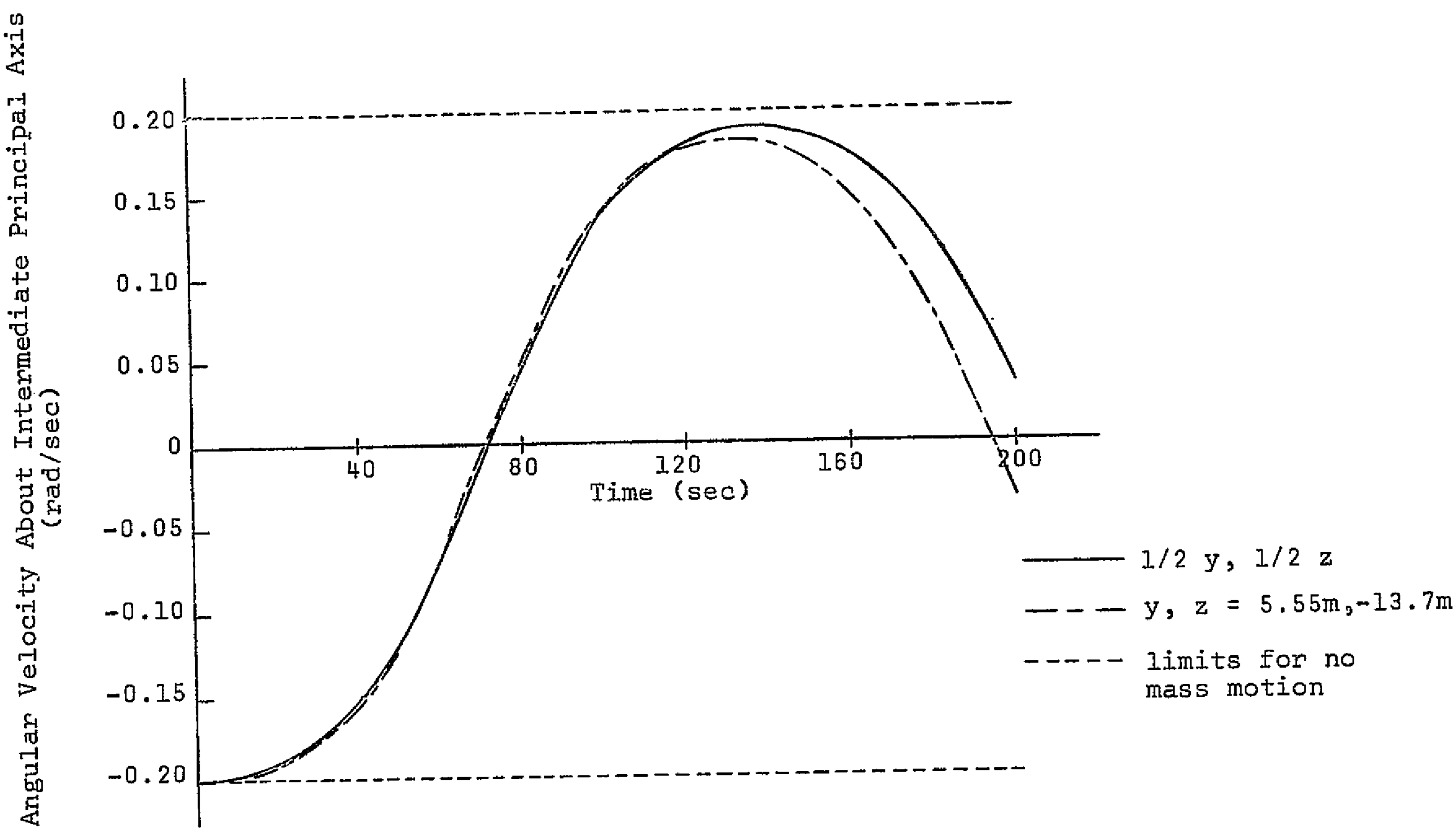

Figure 10. ω_2 Oscillations Using One Percent Mass with Various Values for y and z

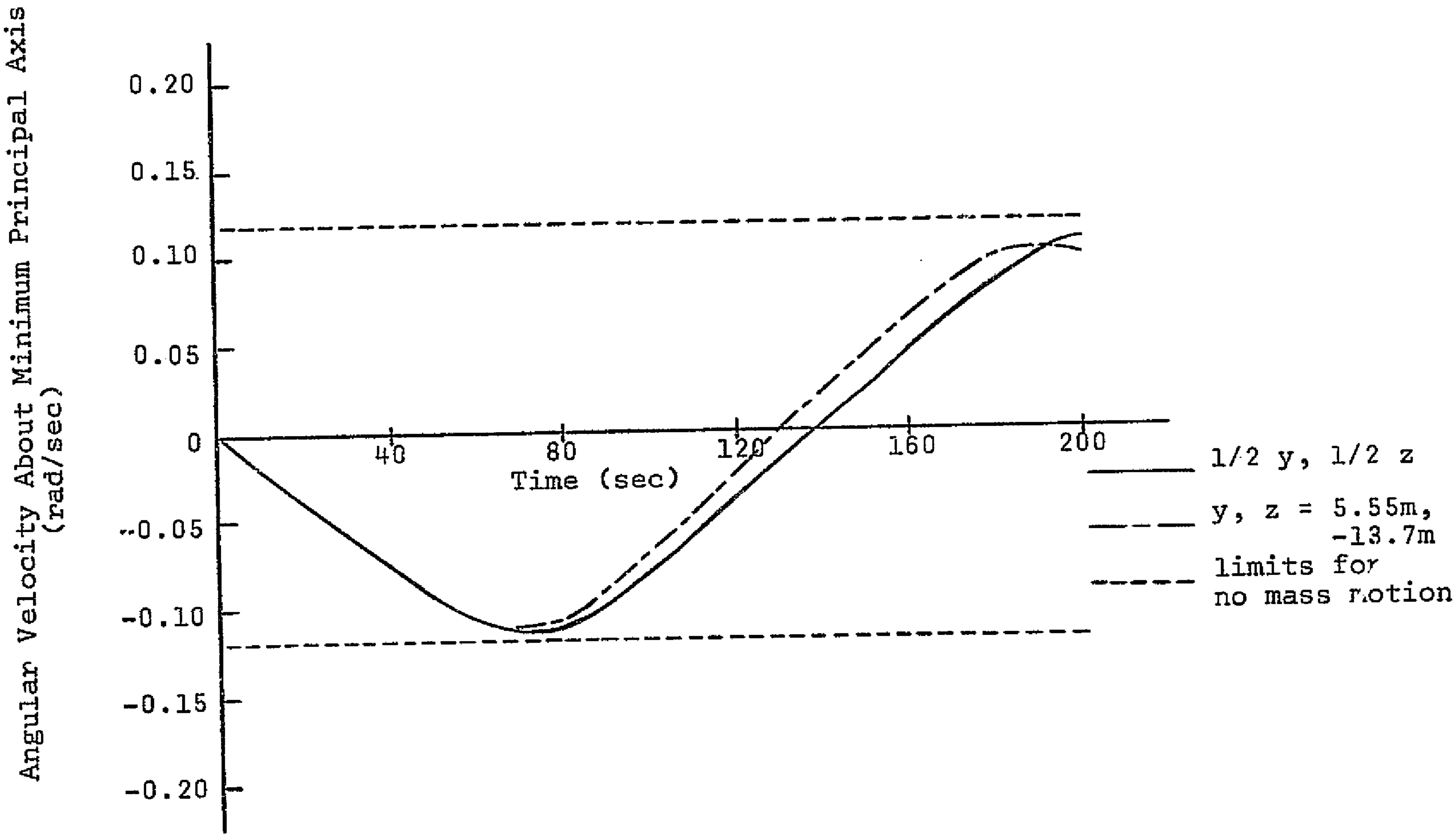

Figure 11. ω_3 Oscillations Using One Percent Mass with Various Values for y and z

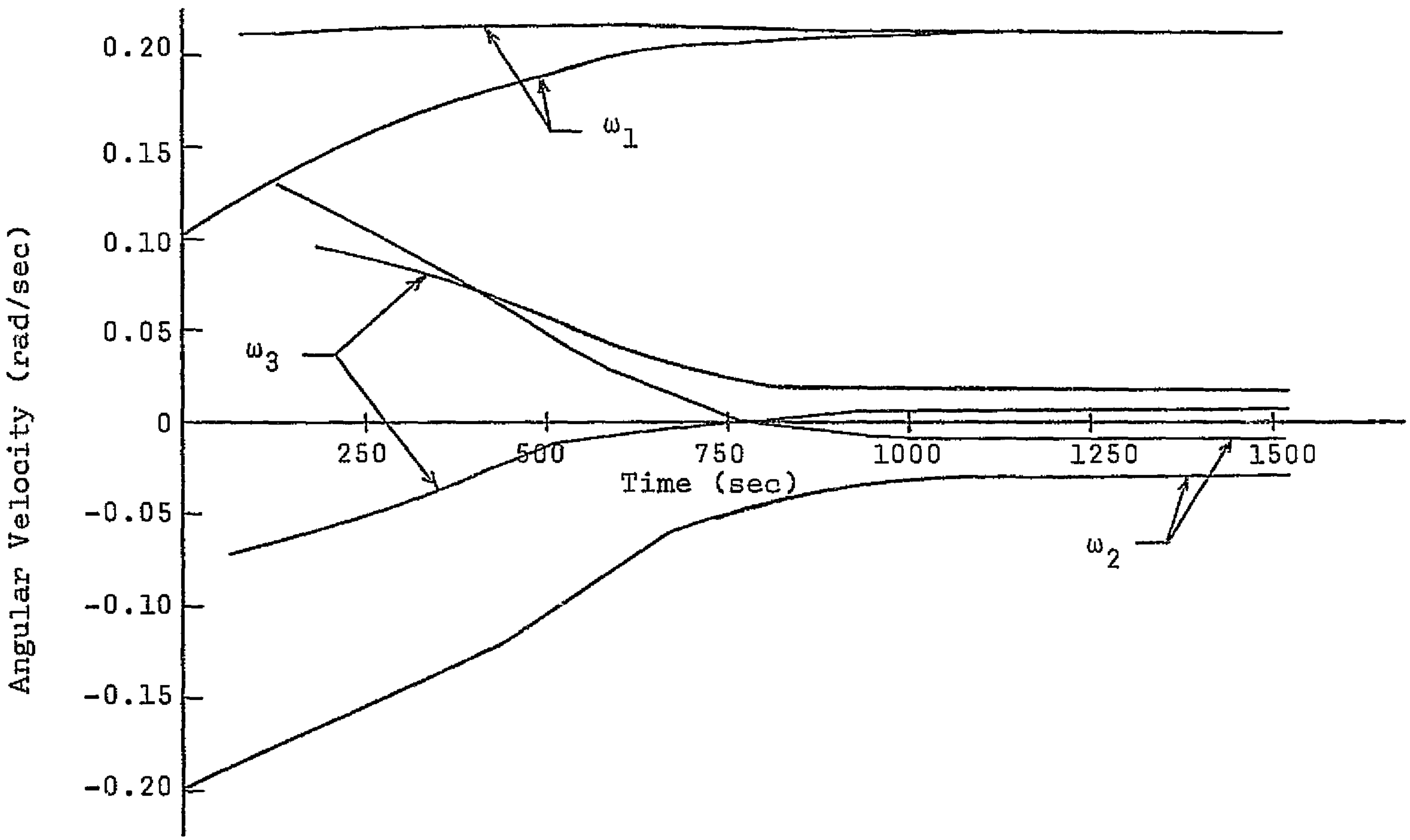

Figure 12. Envelopes of Angular Velocity Oscillations Using One Percent Mass Motion About +10 m

to and have the same sense as the intermediate and minimum inertia axis. Initial values for ω_1, ω_2, and ω_3 are still 0.103 rad/sec, -0.199 rad/sec, and 0.000286 rad/sec, respectively. For both the ω_2 and ω_3 curves, it is evident that one side of the oscillation is decreased faster than the other. One side of the envelope crosses the time axis and, along with the other side, tends to a small, but finite, value. The ω_2 and ω_3 oscillations, then, will not be completely zeroed out; but, they will be made quite small. Comparison to the case shown in Figure 7 which differs only in that the mass oscillation is about the zero x position instead of 10 m, shows that the higher x values initially permit a much faster decrease in the ω_2 and ω_3 envelopes of oscillation. However, having the control mass move through the zero position on the x axis, results in the ω_2 and ω_3 oscillations tending to zero rather than a finite value.

The movement of the control mass parallel to an axis other than that of maximum inertia was investigated by having the mass move parallel to the intermediate inertia axis. This choice of axis and that of the other parameters was made in order to let this case be as similar as possible to the initial one percent mass case; this permits a more valid observation of the effect of the change in the direction of control mass motion relative to the main vehicle. The problem arises since a change in the x direction of mass motion changes the y and z values of the mass and what they represent relative to the main vehicle. Therefore, for this case under consideration, a 998 kg mass was permitted to move 3.7 m about the zero x position. The x axis, along which the control mass moves, was placed parallel to and has the same sense as the intermediate inertia axis. The y and z axes were placed parallel to and have the same sense as the minimum and maximum inertia axes. The y and z positions of the mass were set at -13.7 m and 5.55 m. Initial values of ω_1, ω_2, and ω_3 remain at 0.103 rad/sec, -0.199 rad/sec, and 0.000286 rad/sec. Oscillation envelopes for ω_1, ω_2, and ω_3 are shown in Figure 13. The ω_2 envelope tends to zero. However, the ω_3 envelope is tending to a small, but finite, value. Comparing to the case shown in Figure 7 it is seen that a control mass moving parallel to the maximum inertia axis permits, in addition to zero values for both the ω_2 and ω_3 envelopes, faster detumbling.

These results were obtained by running the comptuer program for simulation times of 100 sec. To bring the extreme mass positions to the permitted magnitudes, in this 100 sec simulation time period, required up to about 35 iterations which used about 100 sec of IBM 370/165 computer time. At the end of each 100 sec simulation

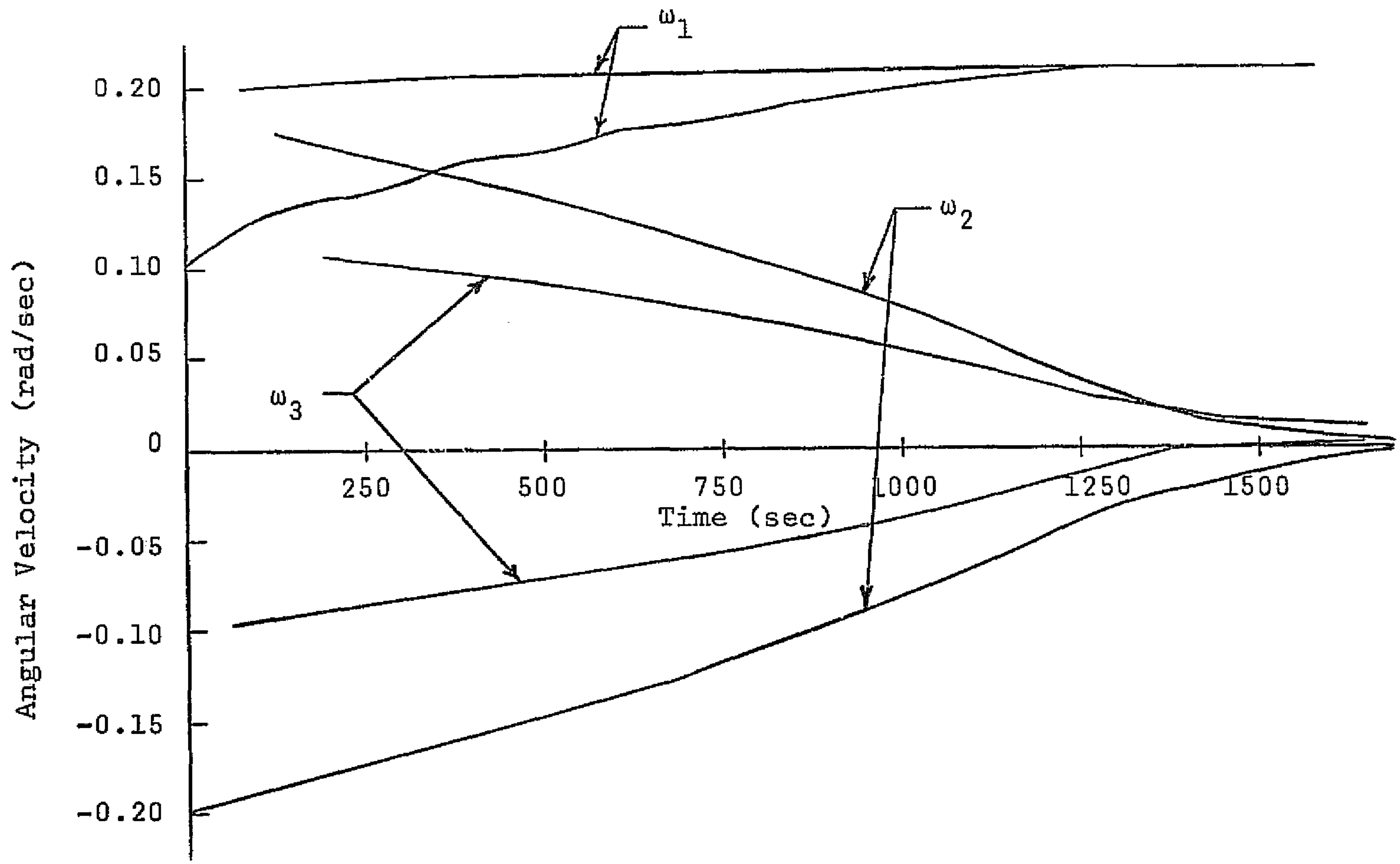

Figure 13. Envelopes of Angular Velocity Oscillations Using One Percent Mass Motion Parallel to the Intermediate Inertia Axis

run the end values for ω_1, ω_2, ω_3, x and $\dot{x}$ were used as initial conditions for the next 100 sec simulation run. This time increment of 100 sec for simulation was chosen since larger time increments made it difficult to bring the extreme mass positions to within the specific limits. A few times even this time increment was too large; that is, the mass would initially stay within the limits but would then move past, where about 35 iterations were the maximum permitted. Rather than extend the computer time and thereby increase the number of iterations, the acceptable initial part of the run was used. For zeroing out the mass position, velocity, and acceleration in the 0.5% case that was initially investigated, the simulation time increment was arbitrarily chosen at 50 sec. Also, the position limits, as stated previously, were set at $\pm 10^{-9}$ m for the zeroing out simulation time. Normally, limits were set at ±2.5 m at the beginning of a case and then changed to about ±3.0 m or higher. As stated previously in the analytical study on the optimization technique, the limits of mass position should be set lower than what is desired since the penalty function comes in when there is a violation of the set limits. The iteration that was chosen during each 100 sec simulation time run had the lowest peak magnitudes for the ω_2 and ω_3 oscillations for mass positions within the ±3.7 m prescribed extreme limits. There was no need to place constraints in the optimization technique on mass velocity and acceleration since, as discussed in the initial 0.5% mass case, these variables did not reach excessive magnitudes. The values of other constants, associated with the optimization method, which were discussed in the analytical study are given in the main program of the computer program listed in Appendix B. The time steps in the integrations were set at 5.0 sec of simulation time.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

An optimal movable mass control system has been applied to a tumbling spacecraft in order to obtain simple spin about the major principal axis. The results indicate that the largest possible magnitudes for the internal mass, length of the linear track, and positions of the mass on the y and z axis will yield the fastest detumbling times. The choice of these values depends upon size and mass of the spacecraft. Results also indicate that the mass should oscillate, about a zero point, on a line parallel to the maximum inertia axis. These results were based on worst case initial conditions for ω_1, ω_2, and ω_3. Tumbling situations that might be encountered in actual space operations will usually be less severe and, therefore, will probably require less time to reach simple spin. Also, these results were based on one vehicle, the modular space station. However, since this vehicle was asymmetric, the optimization technique will apply to any type of spacecraft. The results of various parameter changes, furthermore, were based on motions of a one percent mass and the resultant effects on the peaks of ω_1, ω_2, and ω_3; no further comments were made about $\dot{T}$, f_x, $\dot{x}$, and $\ddot{x}$. A one percent mass was used to show the effects of changes of various parameters since the large mass made the effects readily apparent and showed them in a faster time, compared to smaller masses which may be more feasible for the space station due to its considerable mass. The graphs of ω_1, ω_2, and ω_3 were used since the objective is to reduce the peaks of ω_2 and ω_3, with ω_1 tending to one value. Other variables were not discussed since their behavior and magnitudes were comparable to the 0.5% mass case which was examined in detail. This 0.5% mass case showed that the velocity and acceleration of the mass, and the power requirement are low. Therefore, the use of the optimal control system in actual operations is feasible. Compared to the force control law method, detumbling can be achieved in one-fourth the time. This decrease is considerable since stabilization may require hours. It should be noted that the optimization technique need not only be considered from the standpoint of minimizing time to detumble. Since time increases as mass decreases, a minimum mass solution can be obtained by fixing the time at the largest feasible value. Viewing the comparison to the force control law method in relation to this examination of mass and time changes, it is possible that an object with a mass much smaller than that used in the force control law technique may be used to achieve simple spin in the same time period. When dealing with a mass of about 1,000 kg, this decrease will be quite considerable. This idea of mass decrease for equivalent times by using the optimization technique being investigated here can be applied to wobble damping discussed in Chapter II. Since the angular velocities to be reduced are quite small, time may not be the critical variable. However, mass is always of importance in space applications due to cost per mass to be placed in orbit. The optimization technique will reduce considerably the mass of the movable object needed to achieve simple spin in the same time period of active control.

In regard to the nature of the optimal solutions that were obtained, the local minimum achieved here permits a faster detumbling time or a smaller mass when compared to existing solutions. In addition, the minimum seems to be an absolute minimum since the slopes of the ω_2 and ω_3 envelopes of oscillation for the 0.5% case shown in Figure 3 and the one percent case shown in Figure 7 are approximately constants. Specifically, noting that the constant slopes of these cases are comprised of about 100 sec simulation time increments and that various guesses of the control variable were used, it becomes evident that there is a definite unique decrease in the peaks of the ω_2 and ω_3 oscillations for each case. There seems to be only one minimum for a specific case; hence, it is an absolute minimum.

Considering actual operations in space, a hybrid computer may be more effective since modeling the vehicle dynamics on an analog will permit faster computation; as was discussed in the previous chapter, 100 sec of digital computer time are needed to obtain 100 sec of simulation control time. Comparison between computer predicted vehicle motions due to mass movement based on the optimization method and actual motions could then be continuously monitored, thereby permitting updating of the initial conditions for ω_1, ω_2, and ω_3 for the subsequent simulation time increments for optimization. This updating could be done without having to stop active control.

No attempt was made to improve on the first-order gradient solution; that is, an optimization method such as the neighboring extremal, which would utilize and require the solutions already obtained, was not used to further reduce the time or mass. If a digital system similar to the one used in this investigation was employed, the added computation time would necessitate periods of no active control since the total computer time would be greater than the simulation time. Even if a hybrid system could be used to sufficiently reduce the time needed for the added computation, the decrease in time or mass would probably be slight compared to the quantities already required.

The optimal control was applied specifically to stabilize a tumbling vehicle about its major principal axis. No computer runs were made to achieve simple spin about a specific geometric axis. However, the technique presented here can probably be applied to such a case by appropriately changing the performance index and the direction of mass motion. An indication of this is given by noting that the initial one percent mass case was actually made to spin about an axis approximately ten degrees from the maximum inertia axis of the whole system. This was due to the fact that the major principal axis of the main vehicle was chosen as the direction of the linear track and that the mass is large and far from the center of mass of the main body. In actual operations with large masses, the major principal axis of the system should be used as the direction for the linear track; this was not done in this study since a close comparison to the smaller mass case was desired. To have spin about some geometric axes may, however, necessitate considerably higher energy input by the control system. This will probably occur for spin about axes close to the minimum inertia axis.

It was assumed that the principal moments of inertia of the main body do not change. However, an explosion may result in part of the spacecraft blowing off. Also, tumbling itself may cause loss of part of the vehicle.

If this mass loss results in a large change in the moments of inertia, it should be included in the optimization. Specifically, the magnitudes of the moments of inertia should be corrected and the direction of the linear track should be altered to correspond with the new major principal axis if spin is desired about this axis. If direction change is not feasible, a redefinition of the performance index could be studied; but, as was shown, this would result in a longer time or a larger mass, and residual transeverse angular velocity.

Other types of control mass motions could be investigated using the optimization method presented here. Specifically, a translation other than linear or a rotation relative to the spacecraft.

The optimal control technique investigated is significant in that it uses an open loop solution to control a vehicle in real time regardless of initial conditions. The highly nonlinear equations of motion preclude the use of an optimal closed loop approach. Normally, a non optimal feedback method like the one proposed by Edwards would have to be used. The use of such optimal open loop solutions should be investigated for other active control devices when optimization is desired. One area of application is for minimum time thrusting of control jets with constraints on three orthogonal components.

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APPENDIX A

EQUATIONS OF MOTION FOR A SPACECRAFT WITH A MOVABLE MASS

The equations of motion are given below. The mass is permitted to move along the x axis which is fixed in the main body.

$$\dot{\omega}_x = [VAA(x^2 CMA1 + x^4 CMA2 + CMA3) +$$
$$VBB(x\ CMB1 + x^2 CMB2 + x^3 CMB3 + CMB4) +$$
$$VCC(x\ CMC1 + x^2 CMC2 + x^3 CMC3 + CMC4)]$$
$$[1.0/|\underline{A}|]$$

$$\dot{\omega}_y = [VAA(x\ CNA1 + x^2 CNA2 + x^3 CNA3 + CNA4) +$$
$$VBB(x\ CNB1 + x^2 CNB2 + CNB3) +$$
$$VCC(x\ CNC1 + x^2 CNC2 + CNC3)][1.0/|\underline{A}|]$$

$$\dot{\omega}_z = [VAA(x\ CPA1 + x^2 CPA2 + x^3 CPA3 + CPA4) +$$
$$VBB(x\ CPB1 + x^2 CPB2 + CPB3) +$$
$$VCC(x\ CPC1 + x^2 CPC2 + CPC3)][1.0/|\underline{A}|]$$

where $$VAA = \omega_y^2\ CWA1 + \omega_z^2\ CWA2 + \omega_x\omega_y(CWA3 + x\ CWA4) +$$
$$\omega_x\omega_z(CWA5 + x\ CWA6) + \omega_y\omega_z\ CWA7 +$$
$$\omega_y\ \dot{x}\ CWA8 + \omega_z\ \dot{x}\ CWA9$$

$$VBB = \omega_x^2(CWB1 + x\ CWB2) + \omega_z^2(CWB3 + x\ CWB4) +$$

$$\omega_x\omega_y\ CWB5 + \omega_x\omega_z(CWB6 + x^2CWB7) +$$

$$\omega_y\omega_z(CWB8 + x\ CWB9) + \omega_y x\dot{x}\ CWB10 +$$

$$\ddot{x}\ CWB11$$

$$VCC = \omega_x^2(CWC1 + x\ CWC2) + \omega_y^2(CWC3 + x\ CWC4) +$$

$$\omega_x\omega_y(CWC5 + x^2CWC6) + \omega_x\omega_z\ CWC7 +$$

$$\omega_y\omega_z(CWC8 + x\ CWC9) + \omega_z x\dot{x}\ CWC10 +$$

$$\ddot{x}\ CWC11$$

$$|\underline{A}| = x^4\mu^2CA3 + x^3CA23 + x^2\mu CA2 + xCA12 + CA1$$

$$CA3 = I_x$$

$$CA23 = -2I_{xy}\mu^2y - 2I_{xz}\mu^2z$$

$$CA2 = I_xI_y + I_xI_z - I_{xy}^2 - I_{xz}^2 + I_x\mu y^2 + I_x\mu z^2 +$$

$$I_y\mu y^2 + I_z\mu z^2 - 2I_{yz}\mu yz$$

$$CA12 = -2I_{xy}I_{yz}\mu z - 2I_{xy}\mu^2yz^2 - 2I_{xy}I_z\mu y -$$

$$2I_{xy}\mu^2y^3 - 2I_{yz}I_{xz}\mu y - 2I_{xz}\mu^2y^2z -$$

$$2I_{xz}I_y\mu z - 2I_{xz}\mu^2z^3$$

$$CA1 = I_xI_yI_z - I_xI_{yz}^2 - I_{xy}^2I_z - I_{xz}^2I_y -$$

$$2I_{xy}I_{xz}I_{yz} + I_xI_y\mu y^2 + I_xI_z\mu z^2 -$$

$$2I_xI_{yz}\mu yz + I_yI_z\mu y^2 + I_yI_z\mu z^2 +$$

$$I_y\mu^2y^4 + I_y\mu^2y^2z^2 + I_z\mu^2y^2z^2 + I_z\mu^2z^4 -$$

$$I_{yz}^2\mu y^2 - I_{yz}^2\mu z^2 - 2I_{yz}\mu^2y^3z -$$

$$2I_{yz}\mu^2yz^3 - I_{xy}I_{xz}\mu yz - I_{xy}^2\mu y^2 -$$

$$I_{xz}I_{xy}\mu yz - I_{xz}^2\mu z^2$$

$CWA1 = I_{yz} + \mu yz$

$CWA2 = -I_{yz} - \mu yz$

$CWA3 = I_{xz}$

$CWA4 = \mu z$

$CWA5 = -I_{xy}$

$CWA6 = -\mu y$

$CWA7 = I_y - I_z - \mu y^2 + \mu z^2$

$CWA8 = 2\mu y$

$CWA9 = 2\mu z$

$CWB1 = -I_{xz}$

$CWB2 = -\mu z$

CWB3 = I_{xz}

CWB4 = μz

CWB5 = $-I_{yz} - \mu yz$

CWB6 = $I_z - I_x - \mu z^2$

CWB7 = μ

CWB8 = I_{xy}

CWB9 = μy

CWB10 = -2μ

CWB11 = $-\mu z$

CWC1 = I_{xy}

CWC2 = μy

CWC3 = $-I_{xy}$

CWC4 = $-\mu y$

CWC5 = $-I_y + I_x + \mu y^2$

CWC6 = $-\mu$

CWC7 = $I_{yz} + \mu yz$

CWC8 = $-I_{xz}$

$$CWC9 = -\mu z$$

$$CWC10 = -2\mu$$

$$CWC11 = +\mu y$$

$$CMA1 = I_y\mu + I_z\mu + \mu^2 y^2 + \mu^2 z^2$$

$$CMA2 = \mu^2$$

$$CMA3 = I_y I_z + I_y \mu y^2 + I_z \mu z^2 - I_{yz}^2 - 2I_{yz}\mu yz$$

$$CMB1 = I_{yz}\mu z + \mu^2 z^2 y + I_z \mu y + \mu^2 y^3$$

$$CMB2 = I_{xy}\mu$$

$$CMB3 = \mu^2 y$$

$$CMB4 = I_{yz}I_{xz} + I_{xz}\mu yz + I_{xy}I_z + I_{xy}\mu y^2$$

$$CMC1 = I_{yz}\mu y + \mu^2 y^2 z + I_y \mu z + \mu^2 z^3$$

$$CMC2 = I_{xz}\mu$$

$$CMC3 = \mu^2 z$$

$$CMC4 = I_{xy}I_{yz} + I_{xy}\mu yz + I_y I_{xz} + I_{xz}\mu z^2$$

$$CNA1 = I_{yz}\mu z + \mu^2 yz^2 + I_z \mu y + \mu^2 y^3$$

$$CNA2 = I_{xy}\mu$$

$$CNA3 = \mu^2 y$$

$$CNA4 = I_{yz}I_{xz} + I_{xz}\mu yz + I_{xy}I_z + I_{xy}\mu y^2$$

$$CNB1 = -2I_{xz}\mu z$$

$$CNB2 = I_x\mu + \mu^2 y^2$$

$$CNB3 = I_xI_z + I_x\mu y^2 + I_z\mu y^2 + \mu^2 y^4 + I_z\mu z^2 + \mu^2 y^2 z^2 - I_{xz}^2$$

$$CNC1 = I_{xz}\mu y + I_{xy}\mu z$$

$$CNC2 = \mu^2 yz$$

$$CNC3 = I_{xz}I_{xy} + I_xI_{yz} + I_x\mu yz + I_{yz}\mu y^2 + \mu^2 y^3 z + I_{yz}\mu z^2 + \mu^2 yz^3$$

$$CPA1 = I_{yz}\mu y + \mu^2 y^2 z + I_y\mu z + \mu^2 z^3$$

$$CPA2 = I_{xz}\mu$$

$$CPA3 = \mu^2 z$$

$$CPA4 = I_{xy}I_{yz} + I_{xy}\mu yz + I_yI_{xz} + I_{xz}\mu z^2$$

$$CPB1 = I_{xz}\mu y + I_{xy}\mu z$$

$$CPB2 = \mu^2 yz$$

$$CPB3 = I_{xz}I_{xy} + I_xI_{yz} + I_x\mu yz + I_{yz}\mu y^2 + \mu^2 y^3 z + I_{yz}\mu z^2 + \mu^2 yz^3$$

$$CPC1 = -2I_{xy}\mu y$$

$$CPC2 = I_x\mu + \mu^2 z^2$$

$$CPC3 = I_x I_y + I_x\mu z^2 + I_y\mu y^2 + \mu^2 y^2 z^2 + I_y\mu z^2 + \mu^2 z^4 - I_{xy}^2.$$

APPENDIX B

COMPUTER PROGRAM

A listing of the first-order gradient optimization program is presented on the following pages. A flow chart is given in Figure 14. The following variables need to be specified for each case:

EX = I_x

EY = I_y

EZ = I_z

EXY = I_{xy}

EXZ = I_{xz}

EYZ = I_{yz}

X2MAX = $\omega_{2_{max}}$

X3MAX = $\omega_{3_{max}}$

X1INV = $\omega_x(t_0)$

X2INV = $\omega_y(t_0)$

X3INV = $\omega_z(t_0)$

X4INV = $x(t_0)$

X5INV = $\dot{x}(t_0) = \beta(t_0)$

YPO = y

ZPO = z

FORDIV =
number of time steps
IORDIV =

TF = t_f

PEXLMH = x_h

PEXLML = x_l

PECXVA = $K_1 = K_2$

W = W

EP = ε

SMAMAS = m

BIGMAS = M

YOU = μ

U = u_1.

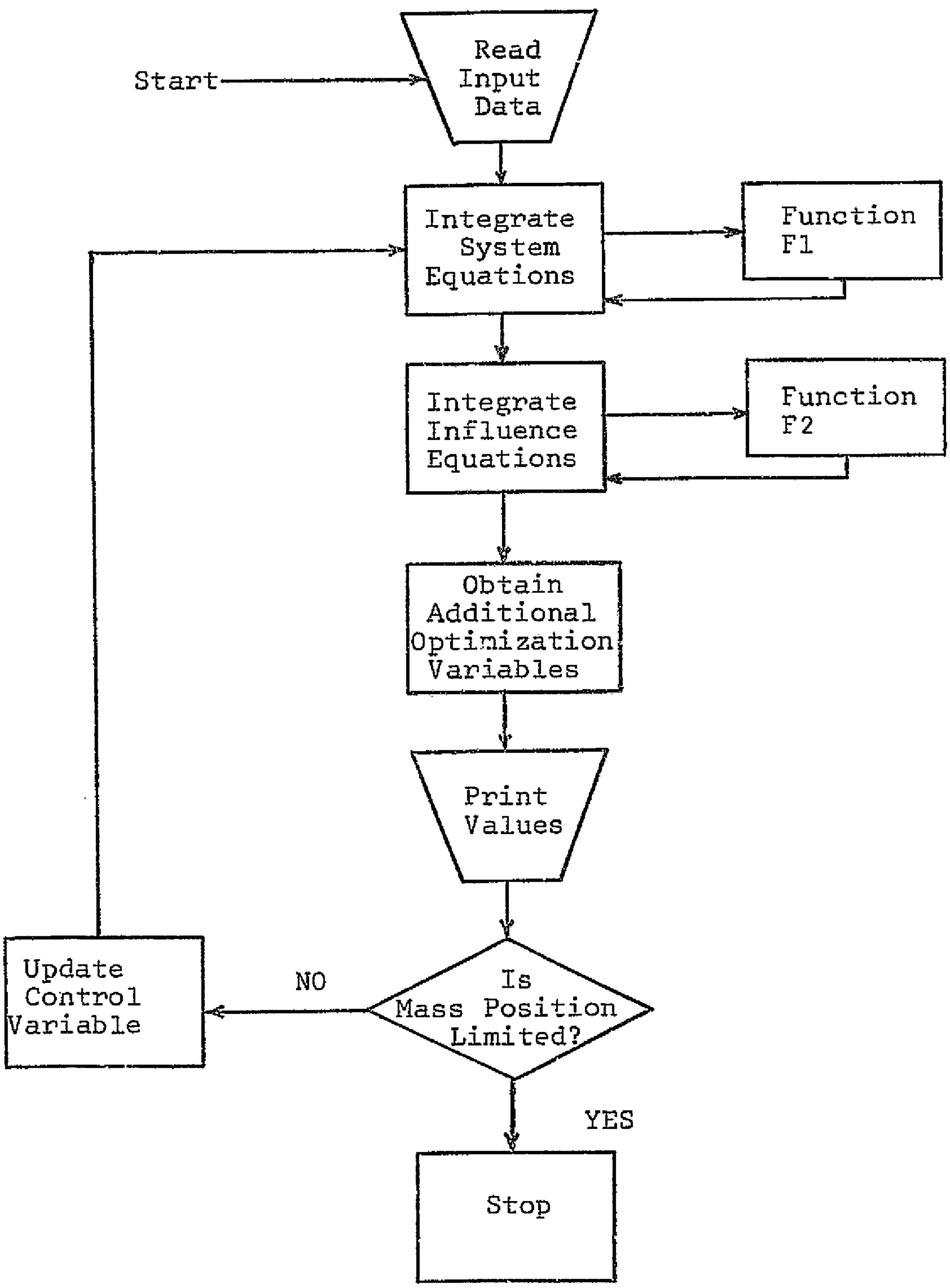

Figure 14. Flow Chart for First-Order Gradient Optimization

```
DIMENSION U(500)
XENDY=+1.00E-20
XENDZ=+1.00E-20
EX=.674209317E+07
EY=.62755678E+07
EZ=.51527998E+07
EXY=0.0
EXZ=0.0
EYZ=0.0
EYX=EXY
EZX=EXZ
EZY=EYZ
X2MAX=.001
X3MAX=.001
SOMC0A=0.0
SOMC1B=1.0
SOMC2C=0.0
X5MAX=1.0
X1INV=.103
X2INV=-.199
X3INV=.000286
X4INV=0.0
X5INV=0.0
P1INV=0.0
P2INV=0.0
P3INV=0.0
P4INV=0.0
P5INV=0.0
WANVAI=10.0
YPO=5.55
ZPO=-13.7
FORDIV=20.0
IORDIV=20
TF=100.0
PEXLIM=2.0
```

```
H=2.5
L=-2.5
A=100.0
0-

A=0.0
M=19.0
S=498.960
S=99791.998
MAMAS*BIGMAS/(SMAMAS+BIGMAS)
D=IORDIV+1
88 KK=1,IORDAD
=.002
NUE
       XYZ(XENDY,XENDZ,EZ,EY,EX,EYZ,EXZ,EXY,EYX,EZX,EZY,W,EP,
H,PEXLML,
,X3MAX,SOMCOA,SOMC1B,SOMC2C,X5MAX,WANVAI,YPO,ZPO,FORDIV,
V,TF,X1INV,X2INV,X3INV,X4INV,X5INV,P1INV,P2INV,P3INV,P4INV,
,PEXLIM,PECXVA,PECUVA,PEULIM,SMAMAS,BIGMAS,YOU,IORDAD,U)




UTINE  XYZ(XENDY,XENDZ,EZ,EY,EX,EYZ,EXZ,EXY,EYX,EZX,EZY,W,EP,
H,PEXLML,
,X3MAX,SOMCOA,SOMC1B,SOMC2C,X5MAX,WANVAI,YPO,ZPO,FORDIV,
V,TF,X1INV,X2INV,X3INV,X4INV,X5INV,P1INV,P2INV,P3INV,P4INV,
,PEXLIM,PECXVA,PECUVA,PEULIM,SMAMAS,BIGMAS,YOU,IORDAD,U)
SION X(7), R(7,4), XB(7), XB1(7), XSTOR1(500), XSTOR2(500),
3(500), XSTOR4(500), U(500), DXSTR1(500), DXSTR2(500),
3(500), DXSTR4(500), TEMPOR(4), TIME(500), P(5), RP(5,4),
, PB1(5), PSTOR1(500), PSTOR2(500), PSTOR3(500), PSTOR4(500),
1(500,2), RSTOR2(500,2), RSTOR3(500,2), RSTOR4(500,2),
,2), AIJ(2), AIIIN1(500), AIIIN2(500), AIIIN3(500),
4(500), AIJIN1(500), AIJIN2(500), AJJINT(500), PAF1U(500),
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7PAF2U(500), PAF3U(500), PAF5U(500), PAF6U(500), V(2), DELTAU(500),
8XSTOR5(500), PSTOR5(500), RSTOR5(500,2)
 CA1=EX*EY*EZ-EX*EYZ**2-EZ*EXY**2-EY*EXZ**2-2.0*EXY*EXZ*EYZ     +
1EX*EY*YOU*YPO**2+EX*EZ*YOU*ZPO**2-2.0*EX*EYZ*YOU*YPO*ZPO+EY*EZ*
2YOU*(YPO**2+ZPO**2)+EY*YOU**2*(YPO**4+YPO**2*ZPO**2)+EZ*YOU**2*
3(YPO**2*ZPO**2+ZPO**4)-EYZ**2*YOU*(YPO**2+ZPO**2)-2.0*EYZ*YOU**2
4*(YPO**3*ZPO+YPO*ZPO**3)-2.0*EXY*EXZ*YOU*YPO*ZPO-EXY**2*YOU*YPO
5**2-EXZ**2*YOU*ZPO**2
 CA2=EX*EY+EX*EZ-EXY**2-EXZ**2                                   +
1EX*YOU*YPO**2+EX*YOU*ZPO**2+EY*YOU*YPO**2+EZ*YOU*ZPO**2-2.0*EYZ
2*YOU*YPO*ZPO
 CA3=EX
 CA23=-2.0*EXY*YOU**2*YPO-2.0*EXZ*YOU**2*ZPO
 CA12=-2.0*EXY*(EYZ*YOU*ZPO+YOU**2*YPO*ZPO**2+EZ*YOU*YPO+YOU**2
1*YPO**3)-2.0*EXZ*(EYZ*YOU*YPO+YOU**2*YPO**2*ZPO+EY*YOU*ZPO+YOU
2**2*ZPO**3)
 CWA1=EYZ+YOU*YPO*ZPO
 CWA2=-CWA1
 CWA3=EXZ
 CWA4=YOU*ZPO
 CWA5=-EXY
 CWA6=-YOU*YPO
 CWA7=EY-EZ-YOU*YPO*YPO+YOU*ZPO*ZPO
 CWA8=2.0*YOU*YPO
 CWA9=2.0*YOU*ZPO
 CWB1=-EXZ
 CWB2=-YOU*ZPO
 CWB3=EXZ
 CWB4=YOU*ZPO
 CWB5=-CWA1
 CWB6=EZ-EX-YOU*ZPO*ZPO
 CWB7=YOU
 CWB8=EXY
 CWB9=YOU*YPO
 CWB10=-2.0*YOU
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CWB 11=-YOU*ZPO
CWC 1=EXY
CWC 2=YOU*YPO
CWC 3=-EXY
CWC 4=-YOU*YPO
CWC 5=-EY+EX+YOU*YPO*YPO
CWC 6=-YOU
CWC 7=CWA 1
CWC 8=-EXZ
CWC 9=-YOU*ZPO
CWC 10=-2.0*YOU
CWC 11=YOU*YPO
CMA 1=EY*YOU+EZ*YOU+YOU*YOU*(YPO*YPO+ZPO*ZPO)
CMA 2=YOU*YOU
CMA 3=EY*EZ+EY*YOU*YPO*YPO+EZ*YOU*ZPO*ZPO-EYZ*EYZ-2.0*EYZ*YOU
1*YPO*ZPO
CMB 1=EYZ*YOU*ZPO+YOU*YOU*ZPO*ZPO*YPO+EZ*YOU*YPO+YOU*YOU*YPO**3
CMB 2=EXY*YOU
CMB 3=YOU*YOU*YPO
CMB 4=EYZ*EXZ+EXZ*YOU*YPO*ZPO+EXY*EZ+EXY*YOU*YPO*YPO
CMC 1=EYZ*YOU*YPO+YOU*YOU*YPO*YPO*ZPO+EY*YOU*ZPO+YOU*YOU*ZPO**3
CMC 2=EXZ*YOU
CMC 3=YOU*YOU*ZPO
CMC 4=EXY*EYZ+EXY*YOU*YPO*ZPO+EY*EXZ+EXZ*YOU*ZPO*ZPO
CNA 1=EYZ*YOU*ZPO+YOU*YOU*YPO*ZPO*ZPO+EZ*YOU*YPO+YOU*YOU*YPO**3
CNA 2=EXY*YOU
CNA 3=YOU*YOU*YPO
CNA 4=EYZ*EXZ+EXZ*YOU*YPO*ZPO+EXY*EZ+EXY*YOU*YPO*YPO
CNB 1=-2.0*EXZ*YOU*ZPO
CNB 2=EX*YOU+YOU*YOU*YPO*YPO
CNB 3=EX*EZ+EX*YOU*YPO*YPO+EZ*YOU*YPO*YPO+YOU*YOU*YPO**4
1+EZ*YOU*ZPO*ZPO+YOU*YOU*YPO*YPO*ZPO*ZPO-EXZ*EXZ
CNC 1=EXZ*YOU*YPO+EXY*YOU*ZPO
CNC 2=YOU*YOU*YPO*ZPO
CNC 3=EXZ*EXY+EX*EYZ+EX*YOU*YPO*ZPO+EYZ*YOU*YPO*YPO+YOU*YOU*YPO
```

```
     1**3*ZPO+EYZ*YOU*ZPO*ZPO+YOU*YOU*YPO*ZPO**3
      CPA1=EYZ*YOU*YPO+YOU*YOU*YPO*YPO*ZPO+EY*YOU*ZPO+YOU*YOU*ZPO**3
      CPA2=EXZ*YOU
      CPA3=YOU*YOU*ZPO
      CPA4=EXY*EYZ+EXY*YOU*YPO*ZPO+EY*EXZ+EXZ*YOU*ZPO*ZPO
      CPB1=CNC1
      CPB2=CNC2
      CPB3=CNC3
      CPC1=-2.0*EXY*YOU*YPO
      CPC2=EX*YOU+YOU*YOU*ZPO*ZPO
      CPC3=EX*EY+EX*YOU*ZPO*ZPO+EY*YOU*YPO*YPO+YOU*YOU*YPO*YPO*ZPO*ZPO
     1+EY*YOU*ZPO*ZPO+YOU*YOU*ZPO**4-EXY*EXY
   24 T=0.0
      DELTA=(TF-T)/FORDIV
      DELTA2=DELTA/2.0
      WRITE(6,610)
  610 FORMAT('1','TIME(SEC)',4X,'WX(RAD/SEC)',2X,'WY(RAD/SEC)',2X,
     1'WZ(RAD/SEC)',2X,'MASSDIS(FT)',2X,'DWX(RD/SC2)',2X,'DWY(RD/SC2)'
     2,2X,'DWZ(RD/SC2)',2X,'DMASDS(F/S)',2X,'U(FT/SC2)')
      X(1)=X1INV
      X(2)=X2INV
      X(3)=X3INV
      X(4)=X4INV
      X(5)=X5INV
      X(6)=0.0
      X(7)=0.0
      XSTOR1(1)=X(1)
      XSTOR2(1)=X(2)
      XSTOR3(1)=X(3)
      XSTOR4(1)=X(4)
      XSTOR5(1)=X(5)
      DO 2 K=1,IORDIV
      DO 3 I=1,7
      R(I,1)=F1(I,T,X,K,U,YOU,     PEXLIM,PECXVA,
     9PEXLMH,PEXLML,
```

```
 ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
 BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
 CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
 DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
 ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
 FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3)
3 XB(I)=X(I)+DELTA2*R(I,1)
  DXSTR1(K)=R(1,1)
  DXSTR2(K)=R(2,1)
  DXSTR3(K)=R(3,1)
  DXSTR4(K)=R(4,1)
  TB=T+DELTA2
  T=T+DELTA
  TIME(1)=0.0
  TIME(K+1)=T
  DO 4 I=1,7
  R(I,2)=F1(I,TB,XB,K,U,YOU,  PEXLIM,PECXVA,
 9PEXLMH,PEXLML,
 ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
 BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
 CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
 DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
 ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
 FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3)
4 XB1(I)=X(I)+DELTA2*R(I,2)
  DO 5 I=1,7
  R(I,3)=F1(I,TB,XB1,K,U,YOU, PEXLIM,PECXVA,
 9PEXLMH,PEXLML,
 ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
 BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
 CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
 DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
 ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
 FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3)
5 XB(I)=X(I)+DELTA*R(I,3)
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```
      DO 6 I=1,7
    6 R(I,4)=F1(I,T,XB,K,U,YOU,    PEXLIM,PECXVA,
     9PEXLMH,PEXLML,
     ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
     BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
     CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
     DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
     ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
     FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3)
      DO 7 I=1,7
    7 X(I)=X(I)+(DELTA/6.0)*(R(I,1)+2.0*(R(I,2)+R(I,3))+R(I,4))
      XSTOR1(K+1)=X(1)
      XSTOR2(K+1)=X(2)
      XSTOR3(K+1)=X(3)
      XSTOR4(K+1)=X(4)
      XSTOR5(K+1)=X(5)
    2 CONTINUE
      PENIT1=X(6)
      PENIT2=X(7)
      K=IORDIV+1
      TEMPOR(1)=XSTOR1(IORDIV+1)
      TEMPOR(2)=XSTOR2(IORDIV+1)
      TEMPOR(3)=XSTOR3(IORDIV+1)
      TEMPOR(4)=XSTOR4(IORDIV+1)
      I=1
      DXSTR1(IORDAD)=F1(I,T,TEMPOR,K,U,YOU, PEXLIM,PECXVA,
     9PEXLMH,PEXLML,
     ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
     BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
     CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
     DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
     ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
     FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3)
      I=2
      DXSTR2(IORDAD)=F1(I,T,TEMPOR,K,U,YOU, PEXLIM,PECXVA,
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     9PEXLMH,PEXLML,
     ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
     BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
     CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
     DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
     ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
     FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3)
      I=3
      DXSTR3(IORDAD)=F1(I,T,TEMPOR,K,U,YOU, PEXLIM,PECXVA,
     9PEXLMH,PEXLML,
     ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
     BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
     CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
     DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
     ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
     FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3)
      I=4
      DXSTR4(IORDAD)=F1(I,T,TEMPOR,K,U,YOU, PEXLIM,PECXVA,
     9PEXLMH,PEXLML,
     ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
     BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
     CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
     DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
     ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
     FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3)
      DO 711 LK=1,IORDAD
      WRITE(6,611) TIME(LK), XSTOR1(LK), XSTOR2(LK), XSTOR3(LK),
     1XSTOR4(LK), DXSTR1(LK), DXSTR2(LK), DXSTR3(LK), XSTOR5(LK), U(LK)
  611 FORMAT('0',10(1X,E12.5))
  711 CONTINUE
      WRITE(6,611) TIME(IORDAD), XSTOR1(IORDAD), XSTOR2(IORDAD),
     1XSTOR3(IORDAD), XSTOR4(IORDAD), DXSTR1(IORDAD), DXSTR2(IORDAD),
     2DXSTR3(IORDAD), XSTOR5(IORDAD), U(IORDAD)
      WRITE(6,612) TF
  612 FORMAT('0',F15.6)
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```
      WRITE(6,7005) PENIT1,PENIT2
 7005 FORMAT('0',2(3X,E11.4))
      DO 4441 KIK=1,IORDAD
      FORCEX=YOU*(U(KIK)+YPO*XSTOR1(KIK)*XSTOR2(KIK)-XSTOR4(KIK)*
     1XSTOR2(KIK)*XSTOR2(KIK)-XSTOR4(KIK)*XSTOR3(KIK)*XSTOR3(KIK)
     2+ZPO*XSTOR1(KIK)*XSTOR3(KIK)+ZPO*DXSTR2(KIK)-YPO*DXSTR3(KIK))
      TDOT=FORCEX*XSTOR5(KIK)
      RINXDN=XSTOR5(KIK)+ZPO*XSTOR2(KIK)-YPO*XSTOR3(KIK)
      RINYDN=XSTOR4(KIK)*XSTOR3(KIK)-ZPO*XSTOR1(KIK)
      RINZDN=YPO*XSTOR1(KIK)-XSTOR4(KIK)*XSTOR2(KIK)
      ENEKIN=.5*(EX*XSTOR1(KIK)*XSTOR1(KIK)+EY*XSTOR2(KIK)*XSTOR2(KIK)+
     1EZ*XSTOR3(KIK)*XSTOR3(KIK))+.5*YOU*(RINXDN*RINXDN+RINYDN*RINYDN+
     2RINZDN*RINZDN)   -XSTOR1(KIK)*XSTOR3(KIK)*EXZ-XSTOR2(KIK)*
     3XSTOR3(KIK)*EYZ-XSTOR1(KIK)*XSTOR2(KIK)*EXY
      ANGMOX=EX*XSTOR1(KIK)-EXY*XSTOR2(KIK)-EXZ*XSTOR3(KIK)+YOU*(YPO*
     1RINZDN-ZPO*RINYDN)
      ANGMOY=-EXY*XSTOR1(KIK)+EY*XSTOR2(KIK)-EYZ*XSTOR3(KIK)+YOU*(ZPO*
     1RINXDN-XSTOR4(KIK)*RINZDN)
      ANGMOZ=-EXZ*XSTOR1(KIK)-EYZ*XSTOR2(KIK)+EZ*XSTOR3(KIK)+YOU*(
     1XSTOR4(KIK)*RINYDN-YPO*RINXDN)
      ANGMOT=SQRT(ANGMOX*ANGMOX+ANGMOY*ANGMOY+ANGMOZ*ANGMOZ)
      WRITE(6,4442) TIME(KIK),FORCEX,TDOT,ENEKIN,ANGMOT,ANGMOX,ANGMOY,
     1ANGMOZ
 4442 FORMAT('0',8(1X,E12.5))
 4441 CONTINUE
      TP=TF
      TFP=T
      PDELTA=-DELTA
      PDELT2=PDELTA/2.0
      DO 14 M=1,3
      GO TO (15,16,20), M
   15 P(1)=P1INV
      P(2)=P2INV
      P(3)=P3INV
      P(4)=P4INV
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      P(5)=P5INV
      PSTOR1(IORDIV+1)=P(1)
      PSTOR2(IORDIV+1)=P(2)
      PSTOR3(IORDIV+1)=P(3)
      PSTOR4(IORDIV+1)=P(4)
      PSTOR5(IORDIV+1)=P(5)
      SOMTC0=SOMC0A
      SOMTC1=SOMC1B
      SOMTC2=SOMC2C
      GO TO 17
   16 P(1)=0.0
      P(2)=0.0
      P(3)=0.0
      P(4)=0.0
      P(5)=0.0
      RSTOR1(IORDIV+1,1)=P(1)
      RSTOR2(IORDIV+1,1)=P(2)
      RSTOR3(IORDIV+1,1)=P(3)
      RSTOR4(IORDIV+1,1)=P(4)
      RSTOR5(IORDIV+1,1)=P(5)
      SOMTC0=0.0
      SOMTC1=0.0
      SOMTC2=0.0
      GO TO 17
   20 P(1)=0.0
      P(2)=0.0
      P(3)=0.0
      P(4)=0.0
      P(5)=0.0
      RSTOR1(IORDIV+1,2)=P(1)
      RSTOR2(IORDIV+1,2)=P(2)
      RSTOR3(IORDIV+1,2)=P(3)
      RSTOR4(IORDIV+1,2)=P(4)
      RSTOR5(IORDIV+1,2)=P(5)
      SOMTC0=0.0
```

```
      SOMTC1=0.0
      SOMTC2=0.0
   17 DO 8 J=1, IORDIV
      DO 9 L=1,5
      RP(L,1)=F2(L,TP,P,SOMTCO,J,U,YOU,XSTOR5,          M,
     9PEXLMH,PEXLML,
     ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
     BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
     CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
     DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
     ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
     FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3,
     1SOMTC1,
     2SOMTC2,
     4XSTOR1, XSTOR2, XSTOR3, XSTOR4, IORDIV, PEXLIM, PECXVA)
    9 PB(L)=P(L)+PDELT2*RP(L,1)
      TPB=TP+PDELT2
      TP=TP+PDELTA
      DO 10 L=1,5
      RP(L,2)=F2(L,TPB,PB,SOMTCO,J,U,YOU,XSTOR5,        M,
     9PEXLMH,PEXLML,
     ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
     BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
     CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
     DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
     ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
     FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3,
     1SOMTC1,
     2SOMTC2,
     4XSTOR1, XSTOR2, XSTOR3, XSTOR4, IORDIV, PEXLIM, PECXVA)
   10 PB1(L)=P(L)+PDELT2*RP(L,2)
      DO 11 L=1,5
      RP(L,3)=F2(L,TPB,PB1,SOMTCO,J,U,YOU,XSTOR5,      M,
     9PEXLMH,PEXLML,
     ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
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```
     BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
     CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
     DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
     ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
     FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3,
     1SOMTC1,
     2SOMTC2,
     4XSTOR1, XSTOR2, XSTOR3, XSTOR4, IORDIV, PEXLIM, PECXVA)
  11 PB(L)=P(L)+PDELTA*RP(L,3)
      DO 12 L=1,5
  12 RP(L,4)=F2(L,TP,PB,SOMTCO,J,U,YOU,XSTOR5,        M,
     9PEXLMH,PEXLML,
     ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
     BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
     CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
     DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
     ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
     FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3,
     1SOMTC1,
     2SOMTC2,
     4XSTOR1, XSTOR2, XSTOR3, XSTOR4, IORDIV, PEXLIM, PECXVA)
      DO 13 L=1,5
  13 P(L)=P(L)+(PDELTA/6.0)*(RP(L,1)+2.0*(RP(L,2)+RP(L,3))+RP(L,4))
      GO TO (18,19,21),M
  18 PSTOR1(IORDIV+1-J)=P(1)
      PSTOR2(IORDIV+1-J)=P(2)
      PSTOR3(IORDIV+1-J)=P(3)
      PSTOR4(IORDIV+1-J)=P(4)
      PSTOR5(IORDIV+1-J)=P(5)
      GO TO 8
  19 RSTOR1(IORDIV+1-J,1)=P(1)
      RSTOR2(IORDIV+1-J,1)=P(2)
      RSTOR3(IORDIV+1-J,1)=P(3)
      RSTOR4(IORDIV+1-J,1)=P(4)
      RSTOR5(IORDIV+1-J,1)=P(5)
```

```
   GO  TO  8
21 RSTOR1(IORDIV+1-J,2)=P(1)
   RSTOR2(IORDIV+1-J,2)=P(2)
   RSTOR3(IORDIV+1-J,2)=P(3)
   RSTOR4(IORDIV+1-J,2)=P(4)
   RSTOR5(IORDIV+1-J,2)=P(5)
 8 CONTINUE
14 CONTINUE
   DO  22 K=1,IORDAD
   COEFAS=CA1+CA2*XSTOR4(K)**2*YOU+CA3*XSTOR4(K)**4*YOU**2
  1+XSTOR4(K)*CA12+XSTOR4(K)**3*CA23
   IF (ABS(U(K)).LT.PEULIM) GO TO 33
   IF (U(K).GT.0.0) GO TO 34
   IF (U(K).LT.0.0) GO TO 35
33 PENCO2=0.0
   GO  TO  36
34 PENCO2=PECUVA
   GO  TO  36
35 PENCO2=-PECUVA
36 PAF1U(K)=(1.0/COEFAS)*(CWB11*(XSTOR4(K)*CMB1+XSTOR4(K)*XSTOR4(K)
  1*CMB2+XSTOR4(K)**3*CMB3+CMB4)+CWC11*(XSTOR4(K)*CMC1+XSTOR4
  2(K)*XSTOR4(K)*CMC2+XSTOR4(K)**3*CMC3+CMC4))
   PAF2U(K)=(1.0/COEFAS)*(CWB11*(XSTOR4(K)*CNB1+XSTOR4(K)*XSTOR4(K)
  1*CNB2+CNB3)+CWC11*(XSTOR4(K)*CNC1+XSTOR4(K)*XSTOR4(K)*CNC2+CNC3))
   PAF3U(K)=(1.0/COEFAS)*(CWB11*(XSTOR4(K)*CPB1+XSTOR4(K)*XSTOR4(K)*
  2CPB2+CPB3)+CWC11*(XSTOR4(K)*CPC1+XSTOR4(K)*XSTOR4(K)*CPC2+CPC3))
   PAF5U(K)=1.0
   PAF6U(K)=2.0*PENCO2*(ABS(U(K))-PEULIM)
   AIIIN1(K)=((RSTOR1(K,1)*PAF1U(K)+RSTOR2(K,1)*PAF2U(K)+RSTOR3(K,1)
  1*PAF3U(K)+RSTOR5(K,1)*PAF5U(K)+PAF6U(K))**2)/W
   AIIIN2(K)=(RSTOR1(K,1)*PAF1U(K)+RSTOR2(K,1)*PAF2U(K)+RSTOR3(K,1)
  1*PAF3U(K)+RSTOR5(K,1)*PAF5U(K)+PAF6U(K))*(RSTOR1(K,2)*PAF1U(K
  2)+RSTOR2(K,2)*PAF2U(K)+RSTOR3(K,2)*PAF3U(K)+RSTOR5(K,2)*PAF5U(K)
  3+PAF6U(K))/W
   AIIIN4(K)=((RSTOR1(K,2)*PAF1U(K)+RSTOR2(K,2)*PAF2U(K)+RSTOR3(K,2)*
```

```
     1PAF3U(K)+RSTOR5(K,2)*PAF5U(K)+PAF6U(K))**2)/W
      AIJIN1(K)=(PSTOR1(K)*PAF1U(K)+PSTOR2(K)*PAF2U(K)+PSTOR3(K)*
     2PAF3U(K)+PSTOR5(K)*PAF5U(K)+PAF6U(K))*(RSTOR1(K,1)*PAF1U(K)
     3+RSTOR2(K,1)*PAF2U(K)+RSTOR3(K,1)*PAF3U(K)+RSTOR5(K,1)*PAF5U(K)
     4+PAF6U(K))/W
      AIJIN2(K)=(PSTOR1(K)*PAF1U(K)+PSTOR2(K)*PAF2U(K)+PSTOR3(K)*PAF3U(K
     1)+PSTOR5(K)*PAF5U(K)+PAF6U(K))*(RSTOR1(K,2)*PAF1U(K)+RSTOR2(K,2
     2)*PAF2U(K)+RSTOR3(K,2)*PAF3U(K)+RSTOR5(K,2)*PAF5U(K)+PAF6U(K))/W
      AJJINT(K)=((PSTOR1(K)*PAF1U(K)+PSTOR2(K)*PAF2U(K)+PSTOR3(K)*PAF3U
     2(K)+PSTOR5(K)*PAF5U(K)+PAF6U(K))**2)/W
   22 CONTINUE
      AII(1,1)=FNTGRL(IORDIV+1, DELTA, AIIIN1)
      AII(1,2)=FNTGRL(IORDIV+1, DELTA, AIIIN2)
      AII(2,1)=AII(1,2)
      AII(2,2)=FNTGRL(IORDIV+1, DELTA, AIIIN4)
      AIJ(1)  =FNTGRL(IORDIV+1, DELTA, AIJIN1)
      AIJ(2)  =FNTGRL(IORDIV+1, DELTA, AIJIN2)
      AJJ     =FNTGRL(IORDIV+1, DELTA, AJJINT)
      WRITE(6,614) AII(1,1), AII(1,2), AII(2,2), AIJ(1), AIJ(2), AJJ
  614 FORMAT('0',6(5X,E10.3))
      DETMV=AII(1,1)*AII(2,2)-AII(1,2)**2
      DLPSI1=-EP*PENIT1
      DLPSI2=-EP*PENIT2
      V(1)=-(1.0/DETMV)*((AII(2,2))*(DLPSI1+AIJ(1))+(-AII(1,2))*(DLPSI2
     1+AIJ(2)))
      V(2)=-(1.0/DETMV)*((-AII(2,1))*(DLPSI1+AIJ(1))+(AII(1,1))*(DLPSI2+
     1AIJ(2)))
      IF(AII(1,1).GT.0.0) GO TO 7701
      GO TO 7703
 7701 IF(AII(2,2).GT.0.0) GO TO 7702
      DETMV=AII(1,1)
      DLPSI1=-EP*PENIT1
      DLPSI2=0.0
      V(1)=-(1.0/DETMV)*(DLPSI1+AIJ(1))
      V(2)=0.0
```

```
      GO TO 7702
 7703 IF(AII(2,2).GT.0.0) GO TO 7704
      DETMV=0.0
      DLPSI1=0.0
      DLPSI2=0.0
      V(1)=0.0
      V(2)=0.0
      GO TO 7702
 7704 DETMV=AII(2,2)
      DLPSI1=0.0
      DLPSI2=-EP*PENIT2
      V(1)=0.0
      V(2)=-(1.0/DETMV)*(DLPSI2+AIJ(2))
 7702 CONTINUE
      DETIS=AII(1,1)*AII(2,2)-AII(1,2)**2
      GOTVAI=AJJ-(1.0/DETIS)*((AIJ(1)*AII(2,2)-AIJ(2)*AII(2,1))*AIJ(1)+
     1(-AIJ(1)*AII(1,2)+AIJ(2)*AII(1,1))*AIJ(2))
      WRITE(6,361) DETMV, V(1), V(2)
  361 FORMAT('0',4(3X,E10.3))
      IF(ABS(XSTOR2(IORDIV+1)).LT.XENDY) GO TO 25
      GO TO 29
   25 IF(ABS(XSTOR3(IORDIV+1)).LT.XENDZ) GO TO 27
      GO TO 29
   27 IF(ABS(GOTVAI).LT.WANVAI) GO TO 28
   29 CONTINUE
      DO 23 K=1,IORDAD
      DELTAU(K)=PAF1U(K)*(PSTOR1(K)+RSTOR1(K,1)*V(1)+RSTOR1(K,2)*V(2))
     1+PAF2U(K)*(PSTOR2(K)+RSTOR2(K,1)*V(1)+RSTOR2(K,2)*V(2))+PAF3U(K)
     2*(PSTOR3(K)+RSTOR3(K,1)*V(1)+RSTOR3(K,2)*V(2))+PAF5U(K)*(PSTOR5(K)
     3+RSTOR5(K,1)*V(1)+RSTOR5(K,2)*V(2))+PAF6U(K)*(V(1)+V(2))
      U(K)=U(K)-DELTAU(K)/W
   23 CONTINUE
      GO TO 24
   28 RETURN
      END
```

```
      FUNCTION F1(I,T,X,K,U,YOU,   PEXLIM,PECXVA,
     9PEXLMH,PEXLML,
     ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
     BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
     CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
     DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
     ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
     FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3)
      DIMENSION X(7), U(500)
      COEFAS=CA1+CA2*X(4)**2*YOU+CA3*X(4)**4*YOU**2+X(4)*CA12+X(4)**3*
     1CA23
      VAA=X(2)*X(2)*CWA1+X(3)*X(3)*CWA2+X(1)*X(2)*(CWA3+X(4)*CWA4)
     1+X(1)*X(3)*(CWA5+X(4)*CWA6)+X(2)*X(3)*CWA7+X(2)*X(5)*CWA8+X(3)
     2*X(5)*CWA9
      VBB=X(1)*X(1)*(CWB1+X(4)*CWB2)+X(3)*X(3)*(CWB3+X(4)*CWB4)
     1+X(1)*X(2)*CWB5+X(1)*X(3)*(CWB6+X(4)*X(4)*CWB7)+X(2)*X(3)*(CWB8
     2+X(4)*CWB9)+X(2)*(X(4)*X(5)*CWB10)+U(K)*CWB11
      VCC=X(1)*X(1)*(CWC1+X(4)*CWC2)+X(2)*X(2)*(CWC3+X(4)*CWC4)+X(1)*
     1X(2)*(CWC5+X(4)*X(4)*CWC6)+X(1)*X(3)*CWC7+X(2)*X(3)*(CWC8+X(4)*
     2CWC9)+X(3)*(X(4)*X(5)*CWC10)+U(K)*CWC11
      VCMA=X(4)*X(4)*CMA1+X(4)**4*CMA2+CMA3
      VCMB=X(4)*CMB1+X(4)*X(4)*CMP2+X(4)**3*CMB3+CMB4
      VCMC=X(4)*CMC1+X(4)*X(4)*CMC2+X(4)**3*CMC3+CMC4
      VCNA=X(4)*CNA1+X(4)*X(4)*CNA2+X(4)**3*CNA3+CNA4
      VCNB=X(4)*CNB1+X(4)*X(4)*CNB2+CNB3
      VCNC=X(4)*CNC1+X(4)*X(4)*CNC2+CNC3
      VCPA=X(4)*CPA1+X(4)*X(4)*CPA2+X(4)**3*CPA3+CPA4
      VCPB=X(4)*CPB1+X(4)*X(4)*CPB2+CPB3
      VCPC=X(4)*CPC1+X(4)*X(4)*CPC2+CPC3
      GO TO (1,2,3,4,5,6,7),I
    1 F1=(1.0/COEFAS)*(VAA*VCMA+VBB*VCMB+VCC*VCMC)
      RETURN
    2 F1=(1.0/COEFAS)*(VAA*VCNA+VBB*VCNB+VCC*VCNC)
      RETURN
    3 F1=(1.0/COEFAS)*(VAA*VCPA+VBB*VCPB+VCC*VCPC)
```

```
      RETURN
    4 F1=X(5)
      RETURN
    5 F1=U(K)
      RETURN
    6 IF(X(4).LT.PEXLMH) GO TO 7001
      F1=PECXVA*(X(4)-PEXLMH)**2
      GO TO 7002
 7001 F1=0.0
 7002 CONTINUE
      RETURN
    7 IF(X(4).GT.PEXLML) GO TO 7003
      F1=PECXVA*(-X(4)+PEXLML)**2
      GO TO 7004
 7003 F1=0.0
 7004 CONTINUE
      RETURN
      END


      FUNCTION F2(L,TP,P,SOMTCO,J,U,YOU,XSTOR5,          M,
     9PEXLMH,PEXLML,
     ACA1,CA2,CA3,CA23,CA12,CWA1,CWA2,CWA3,CWA4,CWA5,CWA6,CWA7,CWA8,
     BCWA9,CWB1,CWB2,CWB3,CWB4,CWB5,CWB6,CWB7,CWB8,CWB9,CWB10,CWB11,
     CCWC1,CWC2,CWC3,CWC4,CWC5,CWC6,CWC7,CWC8,CWC9,CWC10,CWC11,CMA1,
     DCMA2,CMA3,CMB1,CMB2,CMB3,CMB4,CMC1,CMC2,CMC3,CMC4,CNA1,CNA2,
     ECNA3,CNA4,CNB1,CNB2,CNB3,CNC1,CNC2,CNC3,CPA1,CPA2,CPA3,CPA4,
     FCPB1,CPB2,CPB3,CPC1,CPC2,CPC3,
     1SOMTC1,
     2SOMTC2,
     4XSTOR1, XSTOR2, XSTOR3, XSTOR4, IORDIV, PEXLIM, PECXVA)
      DIMENSION P(5), U(500), XSTOR1(500), XSTOR2(500), XSTOR3(500),
     1XSTOR4(500), XSTOR5(500)
      JJ=IORDIV+2-J
      VCMA=XSTOR4(JJ)*XSTOR4(JJ)*CMA1+XSTOR4(JJ)**4*CMA2+CMA3
```

```
      VCMB=XSTOR4(JJ)*CMB1+XSTOR4(JJ)*XSTOR4(JJ)*CMB2+XSTOR4(JJ)**3
     1*CMB3+CMB4
      VCMC=XSTOR4(JJ)*CMC1+XSTOR4(JJ)*XSTOR4(JJ)*CMC2+XSTOR4(JJ)**3
     1*CMC3+CMC4
      VCNA=XSTOR4(JJ)*CNA1+XSTOR4(JJ)*XSTOR4(JJ)*CNA2+XSTOR4(JJ)**3
     1*CNA3+CNA4
      VCNB=XSTOR4(JJ)*CNB1+XSTOR4(JJ)*XSTOR4(JJ)*CNB2+CNB3
      VCNC=XSTOR4(JJ)*CNC1+XSTOR4(JJ)*XSTOR4(JJ)*CNC2+CNC3
      VCPA=XSTOR4(JJ)*CPA1+XSTOR4(JJ)*XSTOR4(JJ)     *CPA2+XSTOR4(JJ)**3
     1*CPA3+CPA4
      VCPB=XSTOR4(JJ)*CPB1+XSTOR4(JJ)*XSTOR4(JJ)*CPB2+CPB3
      VCPC=XSTOR4(JJ)*CPC1+XSTOR4(JJ)*XSTOR4(JJ)*CPC2+CPC3
      COEFAS=CA1+CA2*XSTOR4(JJ)**2*YOU+CA3*XSTOR4(JJ)**4*YOU**2
     1+XSTOR4(JJ)*CA12+XSTOR4(JJ)**3*CA23
      GO TO (1,2,3,4,5),L
    1 PAF1X1=(1.0/COEFAS)*(VCMA*(XSTOR2(JJ)*(CWA3+XSTOR4(JJ)*CWA4)
     1+XSTOR3(JJ)*(CWA5+XSTOR4(JJ)*CWA6))+VCMB*(2.0*XSTOR1(JJ)*(CWB1
     2+XSTOR4(JJ)*CWB2)+XSTOR2(JJ)*CWB5+XSTOR3(JJ)*(CWB6+XSTOR4(JJ)*
     3XSTOR4(JJ)*CWB7))+VCMC*(2.0*XSTOR1(JJ)*(CWC1+XSTOR4(JJ)*CWC2)
     4+XSTOR2(JJ)*(CWC5+XSTOR4(JJ)*XSTOR4(JJ)*CWC6)+XSTOR3(JJ)*CWC7))
      PAF2X1=(1.0/COEFAS)*(VCNA*(XSTOR2(JJ)*(CWA3+XSTOR4(JJ)*CWA4)
     1+XSTOR3(JJ)*(CWA5+XSTOR4(JJ)*CWA6))+VCNB*(2.0*XSTOR1(JJ)*(CWB1
     2+XSTOR4(JJ)*CWB2)+XSTOR2(JJ)*CWB5+XSTOR3(JJ)*(CWB6+XSTOR4(JJ)*
     3XSTOR4(JJ)*CWB7))+VCNC*(2.0*XSTOR1(JJ)*(CWC1+XSTOR4(JJ)*CWC2)
     4+XSTOR2(JJ)*(CWC5+XSTOR4(JJ)*XSTOR4(JJ)*CWC6)+XSTOR3(JJ)*CWC7))
      PAF3X1=(1.0/COEFAS)*(VCPA*(XSTOR2(JJ)*(CWA3+XSTOR4(JJ)*CWA4)
     1+XSTOR3(JJ)*(CWA5+XSTOR4(JJ)*CWA6))+VCPB*(2.0*XSTOR1(JJ)*(CWB1
     2+XSTOR4(JJ)*CWB2)+XSTOR2(JJ)*CWB5+XSTOR3(JJ)*(CWB6+XSTOR4(JJ)*
     3XSTOR4(JJ)*CWB7))+VCPC*(2.0*XSTOR1(JJ)*(CWC1+XSTOR4(JJ)*CWC2)
     4+XSTOR2(JJ)*(CWC5+XSTOR4(JJ)*XSTOR4(JJ)*CWC6)+XSTOR3(JJ)*CWC7))
       F2=-P(1)*PAF1X1-P(2)*PAF2X1-P(3)*PAF3X1
      RETURN
    2 PAF1X2=(1.0/COEFAS)*(VCMA*(2.0*XSTOR2(JJ)*CWA1+XSTOR1(JJ)*(
     1CWA3+XSTOR4(JJ)*CWA4)+XSTOR3(JJ)*CWA7+XSTOR5(JJ)*CWA8)+VCMB*(
     2XSTOR1(JJ)*CWB5+XSTOR3(JJ)*(CWB8+XSTOR4(JJ)*CWB9)+XSTOR4(JJ)*
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```
     3XSTOR5(JJ)*CWB10)+VCMC*(2.0*XSTOR2(JJ)*(CWC3+XSTOR4(JJ)*CWC4)+
     4XSTOR1(JJ)*(CWC5+XSTOR4(JJ)*XSTOR4(JJ)*CWC6)+XSTOR3(JJ)*(CWC8+
     5XSTOR4(JJ)*CWC9)))
      PAF2X2=(1.0/COEFAS)*(VCNA*(2.0*XSTOR2(JJ)*CWA1+XSTOR1(JJ)*(
     1CWA3+XSTOR4(JJ)*CWA4)+XSTOR3(JJ)*CWA7+XSTOR5(JJ)*CWA8)+VCNB*(
     2XSTOR1(JJ)*CWB5+XSTOR3(JJ)*(CWB8+XSTOR4(JJ)*CWB9)+XSTOR4(JJ)*
     3XSTOR5(JJ)*CWB10)+VCNC*(2.0*XSTOR2(JJ)*(CWC3+XSTOR4(JJ)*CWC4)+
     4XSTOR1(JJ)*(CWC5+XSTOR4(JJ)*XSTOR4(JJ)*CWC6)+XSTOR3(JJ)*(CWC8+
     5XSTOR4(JJ)*CWC9)))
      PAF3X2=(1.0/COEFAS)*(VCPA*(2.0*XSTOR2(JJ)*CWA1+XSTOR1(JJ)*(
     1CWA3+XSTOR4(JJ)*CWA4)+XSTOR3(JJ)*CWA7+XSTOR5(JJ)*CWA8)+VCPB*(
     2XSTOR1(JJ)*CWB5+XSTOR3(JJ)*(CWB8+XSTOR4(JJ)*CWB9)+XSTOR4(JJ)*
     3XSTOR5(JJ)*CWB10)+VCPC*(2.0*XSTOR2(JJ)*(CWC3+XSTOR4(JJ)*CWC4)+
     4XSTOR1(JJ)*(CWC5+XSTOR4(JJ)*XSTOR4(JJ)*CWC6)+XSTOR3(JJ)*(CWC8+
     5XSTOR4(JJ)*CWC9)))
      F2=-P(1)*PAF1X2-P(2)*PAF2X2-P(3)*PAF3X2
     1-SDMTC1*XSTOR2(JJ)/(X2MAX**2)
      RETURN
    3 PAF1X3=(1.0/COEFAS)*(VCMA*(2.0*XSTOR3(JJ)*CWA2+XSTOR1(JJ)*(CWA5
     1+XSTOR4(JJ)*CWA6)+XSTOR2(JJ)*CWA7+XSTOR5(JJ)*CWA9)+VCMB*(2.0*
     2XSTOR3(JJ)*(CWB3+XSTOR4(JJ)*CWB4)+XSTOR1(JJ)*(CWB6+XSTOR4(JJ)*
     3XSTOR4(JJ)*CWB7)+XSTOR2(JJ)*(CWB8+XSTOR4(JJ)*CWB9))+VCMC*(
     4XSTOR1(JJ)*CWC7+XSTOR2(JJ)*(CWC8+XSTOR4(JJ)*CWC9)+XSTOR4(JJ)*
     5XSTOR5(JJ)*CWC10))
      PAF2X3=(1.0/COEFAS)*(VCNA*(2.0*XSTOR3(JJ)*CWA2+XSTOR1(JJ)*(CWA5
     1+XSTOR4(JJ)*CWA6)+XSTOR2(JJ)*CWA7+XSTOR5(JJ)*CWA9)+VCNB*(2.0*
     2XSTOR3(JJ)*(CWB3+XSTOR4(JJ)*CWB4)+XSTOR1(JJ)*(CWB6+XSTOR4(JJ)*
     3XSTOR4(JJ)*CWB7)+XSTOR2(JJ)*(CWB8+XSTOR4(JJ)*CWB9))+VCNC*(
     4XSTOR1(JJ)*CWC7+XSTOR2(JJ)*(CWC8+XSTOR4(JJ)*CWC9)+XSTOR4(JJ)*
     5XSTOR5(JJ)*CWC10))
      PAF3X3=(1.0/COEFAS)*(VCPA*(2.0*XSTOR3(JJ)*CWA2+XSTOR1(JJ)*(CWA5
     1+XSTOR4(JJ)*CWA6)+XSTOR2(JJ)*CWA7+XSTOR5(JJ)*CWA9)+VCPB*(2.0*
     2XSTOR3(JJ)*(CWB3+XSTOR4(JJ)*CWB4)+XSTOR1(JJ)*(CWB6+XSTOR4(JJ)*
     3XSTOR4(JJ)*CWB7)+XSTOR2(JJ)*(CWB8+XSTOR4(JJ)*CWB9))+VCPC*(
     4XSTOR1(JJ)*CWC7+XSTOR2(JJ)*(CWC8+XSTOR4(JJ)*CWC9)+XSTOR4(JJ)*
```

```
     5XSTOR5(JJ)*CWC10))
      F2=-P(1)*PAF1X3-P(2)*PAF2X3-P(3)*PAF3X3
     1-SOMTC1*XSTOR3(JJ)/(X3MAX**2)
      RETURN
    4 IF(XSTOR4(JJ).GE.PEXLMH) GO TO 31
      IF(XSTOR4(JJ).LE.PEXLML) GO TO 32
   30 PENCO1=0.0
      GO TO 33
   31 PENCO1=PECXVA
      GO TO (7501, 33, 7501),M
 7501 PENCO1=0.0
      GO TO 33
   32 PENCO1=-PECXVA
      GO TO (7502, 7502, 33),M
 7502 PENCO1=0.0
   33 DVCMA=2.0*XSTOR4(JJ)*CMA1+4.0*XSTOR4(JJ)**3*CMA2
      DVCMB=CMB1+2.0*XSTOR4(JJ)*CMB2+3.0*XSTOR4(JJ)*XSTOR4(JJ)*CMB3
      DVCMC=CMC1+2.0*XSTOR4(JJ)*CMC2+3.0*XSTOR4(JJ)*XSTOR4(JJ)*CMC3
      DVCNA=CNA1+2.0*XSTOR4(JJ)*CNA2+3.0*XSTOR4(JJ)*XSTOR4(JJ)*CNA3
      DVCNB=CNB1+2.0*XSTOR4(JJ)*CNB2
      DVCNC=CNC1+2.0*XSTOR4(JJ)*CNC2
      DVCPA=CPA1+2.0*XSTOR4(JJ)*CPA2+3.0*XSTOR4(JJ)**2*CPA3
      DVCPB=CPB1+2.0*XSTOR4(JJ)*CPB2
      DVCPC=CPC1+2.0*XSTOR4(JJ)*CPC2
      VAA=XSTOR2(JJ)*XSTOR2(JJ)*CWA1+XSTOR3(JJ)*XSTOR3(JJ)*CWA2+XSTOR1
     1(JJ)*XSTOR2(JJ)*(CWA3+XSTOR4(JJ)*CWA4)+XSTOR1(JJ)*XSTOR3(JJ)
     2*(CWA5+XSTOR4(JJ)*CWA6)+XSTOR2(JJ)*XSTOR3(JJ)*CWA7+XSTOR2(JJ)
     3*XSTOR5(JJ)*CWA8+XSTOR3(JJ)*XSTOR5(JJ)*CWA9
      VBB=XSTOR1(JJ)*XSTOR1(JJ)*(CWB1+XSTOR4(JJ)*CWB2)+XSTOR3(JJ)
     1*XSTOR3(JJ)*(CWB3+XSTOR4(JJ)*CWB4)+XSTOR1(JJ)*XSTOR2(JJ)*CWB5
     2+XSTOR1(JJ)*XSTOR3(JJ)*(CWB6+XSTOR4(JJ)*XSTOR4(JJ)*CWB7)+XSTOR2
     3(JJ)*XSTOR3(JJ)*(CWB8+XSTOR4(JJ)*CWB9)+XSTOR2(JJ)*(XSTOR4(JJ)*
     4XSTOR5(JJ)*CWB10)+U(JJ)*CWB11
      VCC=XSTOR1(JJ)*XSTOR1(JJ)*(CWC1+XSTOR4(JJ)*CWC2)+XSTOR2(JJ)
     1*XSTOR2(JJ)*(CWC3+XSTOR4(JJ)*CWC4)+XSTOR1(JJ)*XSTOR2(JJ)*
```

```
     2(CWC5+XSTOR4(JJ)*XSTOR4(JJ)*CWC6)+XSTOR1(JJ)*XSTOR3(JJ)*CWC7
     3+XSTOR2(JJ)*XSTOR3(JJ)*(CWC8+XSTOR4(JJ)*CWC9)+XSTOR3(JJ)*(XSTOR4
     4(JJ)*XSTOR5(JJ)*CWC10)+U(JJ)*CWC11
      PAF1X4=(1.0/COEFAS)*(VAA*DVCMA+VCMA*(XSTOR1(JJ)*XSTOR2(JJ)*CWA4
     1+XSTOR1(JJ)*XSTOR3(JJ)*CWA6)+VBB*DVCMB+VCMB*(XSTOR1(JJ)*XSTOR1(JJ)
     2*CWB2+XSTOR3(JJ)*XSTOR3(JJ)*CWB4+XSTOR1(JJ)*XSTOR3(JJ)*2.0
     3*XSTOR4(JJ)*CWB7+XSTOR2(JJ)*XSTOR3(JJ)*CWB9+XSTOR2(JJ)*XSTOR5(JJ)
     4*CWB10)+VCC*DVCMC+VCMC*(XSTOR1(JJ)*XSTOR1(JJ)*CWC2+XSTOR2(JJ)
     5*XSTOR2(JJ)*CWC4+XSTOR1(JJ)*XSTOR2(JJ)*2.0*XSTOR4(JJ)*CWC6
     6+XSTOR2(JJ)*XSTOR3(JJ)*CWC9+XSTOR3(JJ)*XSTOR5(JJ)*CWC10))
     7+(VAA*VCMA+VBB*VCMB+VCC*VCMC)*(-1.0/(COEFAS*COEFAS))*(4.0*
     8XSTOR4(JJ)**3*YOU*YOU*CA3+3.0*XSTOR4(JJ)*XSTOR4(JJ)*CA23+2.0
     9*XSTOR4(JJ)*YOU*CA2+CA12)
      PAF2X4=(1.0/COEFAS)*(VAA*DVCNA+VCNA*(XSTOR1(JJ)*XSTOR2(JJ)*CWA4
     1+XSTOR1(JJ)*XSTOR3(JJ)*CWA6)+VBB*DVCNB+VCNB*(XSTOR1(JJ)*XSTOR1(JJ)
     2*CWB2+XSTOR3(JJ)*XSTOR3(JJ)*CWB4+XSTOR1(JJ)*XSTOR3(JJ)*2.0
     3*XSTOR4(JJ)*CWB7+XSTOR2(JJ)*XSTOR3(JJ)*CWB9+XSTOR2(JJ)*XSTOR5(JJ)
     4*CWB10)+VCC*DVCNC+VCNC*(XSTOR1(JJ)*XSTOR1(JJ)*CWC2+XSTOR2(JJ)
     5*XSTOR2(JJ)*CWC4+XSTOR1(JJ)*XSTOR2(JJ)*2.0*XSTOR4(JJ)*CWC6
     6+XSTOR2(JJ)*XSTOR3(JJ)*CWC9+XSTOR3(JJ)*XSTOR5(JJ)*CWC10))
     7+(VAA*VCNA+VBB*VCNB+VCC*VCNC)*(-1.0/(COEFAS*COEFAS))*(4.0*
     8XSTOR4(JJ)**3*YOU*YOU*CA3+3.0*XSTOR4(JJ)*XSTOR4(JJ)*CA23+2.0
     9*XSTOR4(JJ)*YOU*CA2+CA12)
      PAF3X4=(1.0/COEFAS)*(VAA*DVCPA+VCPA*(XSTOR1(JJ)*XSTOR2(JJ)*CWA4
     1+XSTOR1(JJ)*XSTOR3(JJ)*CWA6)+VBB*DVCPB+VCPB*(XSTOR1(JJ)*XSTOR1(JJ)
     2*CWB2+XSTOR3(JJ)*XSTOR3(JJ)*CWB4+XSTOR1(JJ)*XSTOR3(JJ)*2.0
     3*XSTOR4(JJ)*CWB7+XSTOR2(JJ)*XSTOR3(JJ)*CWB9+XSTOR2(JJ)*XSTOR5(JJ)
     4*CWB10)+VCC*DVCPC+VCPC*(XSTOR1(JJ)*XSTOR1(JJ)*CWC2+XSTOR2(JJ)
     5*XSTOR2(JJ)*CWC4+XSTOR1(JJ)*XSTOR2(JJ)*2.0*XSTOR4(JJ)*CWC6
     6+XSTOR2(JJ)*XSTOR3(JJ)*CWC9+XSTOR3(JJ)*XSTOR5(JJ)*CWC10))
     7+(VAA*VCPA+VBB*VCPB+VCC*VCPC)*(-1.0/(COEFAS*COEFAS))*(4.0*
     8XSTOR4(JJ)**3*YOU*YOU*CA3+3.0*XSTOR4(JJ)*XSTOR4(JJ)*CA23+2.0
     9*XSTOR4(JJ)*YOU*CA2+CA12)
      IF(PENCO1.GT.0.0) GO TO 9430
      IF(PENCO1.LT.0.0) GO TO 9431
```

```
9430 CONTINUE
      EXTPEV=XSTOR4(JJ)-PEXLMH
      GO TO 9432
9431 CONTINUE
      EXTPEV=-XSTOR4(JJ)+PEXLML
9432 CONTINUE
      F2=-P(1)*PAF1X4-P(2)*PAF2X4-P(3)*PAF3X4-2.0*PENCO1*EXTPEV-SOMTCO*
     1XSTOR4(JJ)/(PEXLIM**2)
      RETURN
    5 PAF1X5=(1.0/COEFAS)*(VCMA*(XSTOR2(JJ)*CWA8+XSTOR3(JJ)*CWA9)+
     1VCMB*(XSTOR2(JJ)*XSTOR4(JJ)*CWB10)+VCMC*(XSTOR3(JJ)*XSTOR4(JJ)
     2*CWC10))
      PAF2X5=(1.0/COEFAS)*(VCNA*(XSTOR2(JJ)*CWA8+XSTOR3(JJ)*CWA9)+
     1VCNB*(XSTOR2(JJ)*XSTOR4(JJ)*CWB10)+VCNC*(XSTOR3(JJ)*XSTOR4(JJ)
     2*CWC10))
      PAF3X5=(1.0/COEFAS)*(VCPA*(XSTOR2(JJ)*CWA8+XSTOR3(JJ)*CWA9)+
     1VCPB*(XSTOR2(JJ)*XSTOR4(JJ)*CWB10)+VCPC*(XSTOR3(JJ)*XSTOR4(JJ)
     2*CWC10))
      F2=-P(1)*PAF1X5-P(2)*PAF2X5-P(3)*PAF3X5-P(4)
     1-SOMTC2*XSTOR5(JJ)/(X5MAX**2)
      RETURN
      END
```